Asymptotic modelling of time-dependent Signorini problem without friction of linear thin plate

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Abstract

In a recent paper, Chacha and Bensayah [2], after the one of Paumier [7], have studied the asymptotic modelling of Coulomb frictional Signorini static problem of linear plate and nonlinear von Kàrmàn plate. Their main result is that the leading term of the asymptotic expansion is solution of a two-dimensional Signorini problem but without friction. In this paper, we extend this study for dynamical frictionless problem.

Keywords : unilateral contact, Signorini problem, dynamic problem, Coulomb friction.

1 Introduction

The modelling of unilateral contact problems of elastic bodies with friction is quite challenging and encounters some difficulties. The first formulation of the problem (without friction) has been established by Signorini in 1933. Its mathematical analysis is due to Fichera [5] using an equivalent minimizing problem. Some existence results for a class of problems are established by Duvaut and Lions [4] where they have pointed out an open problem of existence and uniqueness in the case of Coulomb friction law (local). In 1980, Néčas, Jarušek and Haslinger [9] have established only the existence of a solution under the condition that the friction coefficient is small enough. After that, more general results have been established by Jarušek [8], Kato [17], Eck and Jarušek [3]. R. Hassani, P.Hild and I.Ionescu [16] have found a sufficient condition for non uniqueness result.

Thin structures are elastic bodies for which one dimension is small compared with other ones, standard exemples are plates, shells and rods. Under realistic mechanical hypotheses, some models are proposed by Kirchhoff, Love, Mindlin, Reissner, Koiter and Naghdi. In 1979, Ciarlet and Destyunder [12] performed a mathematical justification of some of these models. The unilateral contact problem of thin plate with Coulomb friction was treated by Dhia [6] using a penalty method. In 2003, J.C.Paumier [7] has studied the linearized elastostatic Signorini problem with Coulomb friction of a plate where he has proved that the three dimensional problem converges strongly to the solution of the two-dimensional Signorini problem without friction. By mean of formal asymptotic expansions, Chacha and Bensayah [2] have got the same result in the non-linear case (von Kármán equations), this was pointed out by J.C.Paumier as an open problem. Léger and Miara [1] studied a linear frictionless shallow shell by the use of the convergence method.

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The aim of this paper is to extend the work of Paumier [7] to a dynamic state problem but without friction by using at first the formal asymptotic expansion method and then the convergence method. First, we give the strong formulation of the three-dimensional contact problem. Next, we rewrite the problem in a weak form. Using a convenient scaling of the unknowns and data, we get the scaled variational problem. For the first method, called the *displacement-stress approach*, we insert the formal asymptotic expansion of the unknowns in the scaled variational problem. In that way we characterize the problem solved by the leading term (u^0, σ^0) of the expansion. For the second method, called the *convergence method*, we prove that a subsequence of the sequence $(u(\varepsilon), \sigma(\varepsilon))$ has a weak star limit denoted $(u(0), \sigma(0))$ which solves the same two dimensional problem than (u^0, σ^0).

2 Setting of the problem

In this work, we use the following conventions and notations; Greek indices belong to the set $\{1, 2\}$, Latin indices belong to the set $\{1, 2, 3\}$, the symbols of differentiation $\partial_i = \partial/\partial x_i$, $\partial_i^\varepsilon = \partial/\partial x_i^\varepsilon$, $\frac{\partial v}{\partial t} = \dot{v}$, $\frac{\partial^2 v}{\partial t^2} = \ddot{v}$, δ_{ij} the Kronecker symbols, and the summation convention with respect to the repeated indices is used. Let $\omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz boundary γ. We consider a plate as a three-dimensional body, occupying the volume $\Omega^\varepsilon = \omega \times]-\varepsilon, \varepsilon[$, where ε is small parameter $(0 < \varepsilon \leq 1)$. We denote the lateral boundary by $\Gamma_0^\varepsilon = \gamma \times [-\varepsilon, \varepsilon]$, the upper and the lower faces respectively by $\Gamma_+^\varepsilon = \omega \times \{+\varepsilon\}$ and $\Gamma_-^\varepsilon = \omega \times \{-\varepsilon\}$. We denote by $\bar{v}$ the trace of v on Γ_+^ε and by $\underline{v}$ the trace of v on Γ_-^ε, or simply by v if there is no confusion. We restrict ourselves to the case of an isotropic and homogeneous elastic body with Lamé constants $\lambda > 0$, $\mu > 0$ in its natural configuration and having ρ^ε as a volume density. This plate is subjected to body force f^ε on $\Omega^\varepsilon \times]0, +\infty[$ and to surface force g^ε on $\Gamma_-^\varepsilon \times]0, +\infty[$ and is in unilateral contact on Γ_+^ε with a rigid obstacle which occupies the domain $O^\varepsilon = \left\{x^\varepsilon \in \mathbb{R}^3/\,(x_1^\varepsilon, x_2^\varepsilon) \in \omega, x_3^\varepsilon > \varepsilon\right\}$. The contact condition is defined by the inequality $\bar{v}_3 \leq 0$. We assume that this system is in dynamic state and the contact is without friction.

3 Strong and weak Formulation of the problem

Using the above assumptions one can state the following classical elastodynamic Signorini problem without friction:

Find $(u^\varepsilon, \sigma^\varepsilon) := (u^\varepsilon(x^\varepsilon, t), \sigma^\varepsilon(u^\varepsilon))$ such that for all $t \geq 0$:

$$\rho^\varepsilon \frac{\partial^2 u_i^\varepsilon}{\partial t^2} - \partial_j^\varepsilon \sigma_{ij}^\varepsilon = f_i^\varepsilon \; in \; \Omega^\varepsilon \times]0, +\infty[$$
$$\sigma_{ij}^\varepsilon n_j^\varepsilon = g_i^\varepsilon \;\; on\, \Gamma_-^\varepsilon \times]0, +\infty[$$
$$u^\varepsilon = 0 \;\; on\; \Gamma_0^\varepsilon \times]0, +\infty[\,,$$

with Signorini boundary conditions

$$\bar{u}_3^\varepsilon \leq 0, \sigma_{33}^\varepsilon \leq 0, \sigma_{33}^\varepsilon \bar{u}_3^\varepsilon = 0\, on\, \Gamma_+^\varepsilon \times]0, +\infty[\,.$$

The contact is without friction that is interpreted by $\sigma_{\alpha 3}^\varepsilon = 0$ on $\Gamma_+^\varepsilon \times]0, +\infty[$, and finally the initial conditions are

$$u^\varepsilon(.,0) = p^\varepsilon, \dot{u}^\varepsilon(.,0) = q^\varepsilon,$$

where $\sigma_{ij}^\varepsilon(u^\varepsilon) = \lambda e_{pp}^\varepsilon(u^\varepsilon)\delta_{ij} + 2\mu e_{ij}^\varepsilon(u^\varepsilon)$ are the components of the stress tensor, and also represent the constitutive equation of the elastic material, $e_{ij}^\varepsilon(u^\varepsilon) = \frac{1}{2}\left(\partial_i^\varepsilon u_j^\varepsilon + \partial_j^\varepsilon u_i^\varepsilon\right)$ being the components of the linearized deformation tensor.

The vector field $\sigma^\varepsilon_{ij} n^\varepsilon_j$ represents the force acting on the surface section ds whose the unit outward normal vector n^ε. The quantity $\sigma^\varepsilon_N = \sigma^\varepsilon_{ij} n^\varepsilon_j n^\varepsilon_i$ is the component of the pression force and $\sigma^\varepsilon_T = \sigma^\varepsilon n^\varepsilon - \sigma^\varepsilon_N n^\varepsilon$ is the friction force. In our case, $\sigma^\varepsilon_N = \sigma^\varepsilon_{33}$ and $\sigma^\varepsilon_T = (\sigma^\varepsilon_{13}, \sigma^\varepsilon_{23}, 0)$ on Γ^ε_+. Note that we keep the same notation of the function to denote its trace.

We rewrite the above boundary value problem in the following weak form, by using Green's formula, we show that any smooth solution of the boundary value problem also satisfies the following variational problem (VP^ε):

$$
\begin{gathered}
\text{Find } \ (u^\varepsilon(t), \sigma^\varepsilon(t)) \in \vec{K}(\Omega^\varepsilon) \times \mathbb{L}^2_s(\Omega^\varepsilon), \ t \geq 0 \text{ such that} \\
\frac{\partial^2}{\partial t^2} \rho^\varepsilon \int_{\Omega^\varepsilon} u^\varepsilon_i v^\varepsilon_i dx^\varepsilon + a^\varepsilon(u^\varepsilon, v^\varepsilon) = L^\varepsilon(v^\varepsilon) + \langle \sigma^\varepsilon_{33}, \bar{v}^\varepsilon_3 \rangle, \ \forall v^\varepsilon \in \vec{V}(\Omega^\varepsilon), \ t > 0 \\
\langle \sigma^\varepsilon_{33}, v^\varepsilon_3 - u^\varepsilon_3 \rangle \geq 0, \forall v^\varepsilon_3 \in K(\Omega^\varepsilon), t > 0 \\
\sigma^\varepsilon_{ij}(u^\varepsilon) = \lambda e^\varepsilon_{pp}(u^\varepsilon)\delta_{ij} + 2\mu e^\varepsilon_{ij}(u^\varepsilon) \\
u^\varepsilon(.,0) = p^\varepsilon, \dot{u}^\varepsilon(.,0) = q^\varepsilon,
\end{gathered}
$$

where

$$
\begin{aligned}
a^\varepsilon(u^\varepsilon, v^\varepsilon) &= \int_{\Omega^\varepsilon} \sigma^\varepsilon_{ij}(u^\varepsilon) \partial^\varepsilon_j v^\varepsilon_i dx^\varepsilon \\
&= \int_{\Omega^\varepsilon} \left[\lambda e^\varepsilon_{ii}(u^\varepsilon) e^\varepsilon_{jj}(v^\varepsilon) + 2\mu \ e^\varepsilon_{ij}(u^\varepsilon) e^\varepsilon_{ij}(v^\varepsilon) \right] dx^\varepsilon \\
e^\varepsilon_{ij}(u^\varepsilon) &= \frac{1}{2} \left(\partial^\varepsilon_i u^\varepsilon_j + \partial^\varepsilon_j u^\varepsilon_i \right) \\
L^\varepsilon(v^\varepsilon) &= \int_{\Omega^\varepsilon} f^\varepsilon_i v^\varepsilon_i dx^\varepsilon + \int_{\Gamma^\varepsilon_-} g^\varepsilon_i \underline{v}^\varepsilon_i d\Gamma,
\end{aligned}
$$

$$\langle \sigma^\varepsilon_{33}, \phi^\varepsilon_3 \rangle \text{ means the duality pairing on } \Gamma^\varepsilon_+.$$

and

$$
\begin{aligned}
V(\Omega^\varepsilon) &= \left\{ v^\varepsilon \in H^1(\Omega^\varepsilon) / v^\varepsilon = 0 \ on \ \Gamma^\varepsilon_0 \right\}, K(\Omega^\varepsilon) = \{ v^\varepsilon \in \ V(\Omega^\varepsilon) / \ \bar{v}^\varepsilon_3 \leq 0 \}, \\
\vec{V}(\Omega^\varepsilon) &= V(\Omega^\varepsilon) \times V(\Omega^\varepsilon) \times V(\Omega^\varepsilon), \vec{K}(\Omega^\varepsilon) = V(\Omega^\varepsilon) \times V(\Omega^\varepsilon) \times K(\Omega^\varepsilon),
\end{aligned}
$$

$$\mathbb{L}^2_s(\Omega^\varepsilon) = \{ \tau^\varepsilon = (\tau^\varepsilon_{ij}) \in (L^2(\Omega^\varepsilon))^9; \ \tau^\varepsilon_{ij} = \tau^\varepsilon_{ji} \}.$$

As described in [10], under the assumptions $f^\varepsilon_i \in W^{2,\infty}(0,T,L^2(\Omega^\varepsilon))$, $g^\varepsilon_i \in W^{2,\infty}(0,T,L^2(\Gamma^\varepsilon_-) \cap H^{-1/2}(\Gamma^\varepsilon_-))$ and the initial conditions p^ε_i, q^ε_i are in $H^1(\Omega^\varepsilon)$ with $\text{div}\sigma^\varepsilon(p^\varepsilon) \in (L^2(\Omega^\varepsilon))^3$, the problem (VP^ε) admits a solution u^ε verifying $u^\varepsilon \in L^\infty(0,T,\vec{K}(\Omega^\varepsilon))$, $\dot{u}^\varepsilon \in L^\infty(0,T,(L^2(\Omega^\varepsilon))^3)$ and $\ddot{u}^\varepsilon \in \mathcal{D}'(0,T,(L^2(\Omega^\varepsilon))^3)$. The stress tensor $\sigma^\varepsilon(u^\varepsilon)$ belongs to $\mathcal{D}'(0,T,E_{ad}(g^\varepsilon)) \cap L^\infty(0,T,(L^2(\Omega^\varepsilon))^9)$ with $E_{ad}(g^\varepsilon) = \{ \tau^\varepsilon \in \mathbb{L}^2_s(\Omega^\varepsilon); \ \text{div}\tau^\varepsilon \in (L^2(\Omega^\varepsilon))^3, \tau^\varepsilon_{\alpha 3} = 0 \text{ and } \tau^\varepsilon_{33} \leq 0 \text{ on } \Gamma^\varepsilon_+; \tau^\varepsilon n^\varepsilon = g^\varepsilon \text{ on } \Gamma^\varepsilon_- \}$.
The duality pairing $\langle \sigma^\varepsilon_{33}, \phi^\varepsilon_3 \rangle$ on Γ^ε_+ can be expressed as an integral on Γ^ε_+. Indeed, we have $\sigma^\varepsilon_{33} \in L^2(\Omega^\varepsilon)$ and $\partial^\varepsilon_3 \sigma^\varepsilon_{33} \in L^2(\Omega^\varepsilon)$ then $\sigma^\varepsilon_{33} \in L^2(\Gamma^\varepsilon_+)$, in sense of trace, with $||\sigma^\varepsilon_{33}||_{L^2(\Gamma^\varepsilon_+)} \leq C(||\sigma^\varepsilon_{33}||^2_{L^2(\Omega^\varepsilon)} + ||\partial^\varepsilon_3 \sigma^\varepsilon_{33}||^2_{L^2(\Omega^\varepsilon)})^{1/2}$ (see [15] page 219). More general, elements of $H(\text{div}, \Omega^\varepsilon) = \{ \tau^\varepsilon = (\tau^\varepsilon_{ij}) \in L^2(\Omega^\varepsilon); \text{div}\tau^\varepsilon \in (L^2(\Omega^\varepsilon))^3 \}$ have normal traces $\gamma_N(\sigma^\varepsilon)$ on Γ^ε_+.

Remark 3.1. The problem (VP^ε) is *formally* equivalent to the variational inequality: Find a displacement $u^\varepsilon : [0,T] \to \vec{K}(\Omega^\varepsilon)$ such that

$$
\left\{
\begin{array}{c}
\rho^\varepsilon \int_{\Omega^\varepsilon} \ddot{u}^\varepsilon_i (v^\varepsilon_i - \dot{u}^\varepsilon_i) dx^\varepsilon + a^\varepsilon(u^\varepsilon, v^\varepsilon - \dot{u}^\varepsilon) \geq L^\varepsilon(v^\varepsilon - \dot{u}^\varepsilon), \forall v^\varepsilon \in \vec{K}(\Omega^\varepsilon), \ a.e \ t > 0 \\
u^\varepsilon(.,0) = p^\varepsilon, \dot{u}^\varepsilon(.,0) = q^\varepsilon.
\end{array}
\right.
\tag{3.1}
$$

4 Asymptotic study

In this section, we follow [13]. First we transform the shell problem into a scaled problem posed over a set Ω independent of ε. After that, according to the basic Ansatz of the method of formal asymptotic expansions, we inject the formal expansion of the unknowns in the scaled variational problem. Finally, we identify the leading term of the formal expansion of the scaled displacement and the scaled stress tensor.

4.1 Assumptions on data

Let the mapping:

$$\begin{array}{rcl} \chi^{\varepsilon} & : & \Omega \to \Omega^{\varepsilon} \\ (x_1, x_2, x_3) & \to & (x_1^{\varepsilon}, x_2^{\varepsilon}, x_3^{\varepsilon}) \,/ x_1^{\varepsilon} = x_1, x_2^{\varepsilon} = x_2, x_3^{\varepsilon} = \varepsilon x_3 \end{array}$$

hence

$$\begin{array}{rcl} \chi^{\varepsilon}(\Omega) & = & \Omega^{\varepsilon}; \Omega \; = \omega \times]-1, +1[\, ; \chi^{\varepsilon}(\Gamma_{-}^{\varepsilon}) = \Gamma_{-} = \omega \times \{-1\} \\ \chi^{\varepsilon}(\Gamma_{-}^{\varepsilon}) & = & \Gamma_{-} = \omega \times \{+1\}\, ; \; \chi^{\varepsilon}(\Gamma_0^{\varepsilon}) = \Gamma_0 = \partial\omega \times [-1, +1] \end{array}$$

We assume that $\rho^{\varepsilon} = \varepsilon^2 \rho$ and make the change (scaling) of unknowns:

$$\left\{ \begin{array}{c} u_{\alpha}^{\varepsilon} \circ \chi^{\varepsilon} = \varepsilon^2 u_{\alpha}(\varepsilon)\,, \quad u_3^{\varepsilon} \circ \chi^{\varepsilon} = \varepsilon^1 u_3(\varepsilon) \\ v_{\alpha}^{\varepsilon} \circ \chi^{\varepsilon} = \varepsilon^2 v_{\alpha}(\varepsilon)\,, \quad v_3^{\varepsilon} \circ \chi^{\varepsilon} = \varepsilon^1 v_3(\varepsilon) \\ \sigma_{\alpha\beta}(\varepsilon) = \varepsilon^{-2}\; \sigma_{\alpha\beta}^{\varepsilon}(u^{\varepsilon})\,,\; \sigma_{\alpha 3}(\varepsilon) = \varepsilon^{-3} \sigma_{\alpha 3}^{\varepsilon}(u^{\varepsilon})\,,\; \sigma_{33}(\varepsilon) = \varepsilon^{-4}\; \sigma_{33}^{\varepsilon}(u^{\varepsilon}) \end{array} \right. \tag{4.1}$$

The scaling of the contact condition is defined by : $\bar{v}_3 \leq 0$
Then we denote

$$\begin{array}{rcl} V(\Omega) & = & \left\{ v \in H^1(\Omega) / v = 0 \quad \text{on} \;\; \Gamma_0 \right\}, \vec{V}(\Omega) = \; V(\Omega) \times \; V(\Omega) \times \; V(\Omega) \\ K(\Omega) & = & \{ v \in V(\Omega) /\; \bar{v}_3 \leq 0 \quad \text{on} \;\; \Gamma_{+} \}\,,\; \vec{K}(\Omega) = V(\Omega) \times \; V(\Omega) \times K(\Omega) \end{array}$$

$$\mathbb{L}_s^2(\Omega) = \{\tau = (\tau_{ij}) \in (L^2(\Omega))^9;\, \tau_{ij} = \tau_{ji}\}.$$

For the forces we suppose that the following scaling holds: there exists f_i, g_i independent of ε such that

$$\left\{ \begin{array}{l} f_{\alpha}^{\varepsilon} \;\circ \chi^{\varepsilon} = \varepsilon^2 f_{\alpha}, \;\; f_3^{\varepsilon} \;\circ \chi^{\varepsilon} = \varepsilon^3 f_3 \\ g_{\alpha}^{\varepsilon} \;\circ \chi^{\varepsilon} = \varepsilon^3 g_{\alpha}, \;\; g_3^{\varepsilon} \;\circ \chi^{\varepsilon} = \varepsilon^4 g_3 \end{array} \right. \tag{4.2}$$

We finally suppose that

$$p_{\alpha}^{\varepsilon} = \varepsilon^2 p_{\alpha}(\varepsilon), p_3^{\varepsilon} = \varepsilon p_3(\varepsilon), q_{\alpha}^{\varepsilon} = \varepsilon^2 q_{\alpha}(\varepsilon), q_3^{\varepsilon} = \varepsilon q_3(\varepsilon) \tag{4.3}$$

The scaling of the differential operators is clearly governed by

$$\partial_{\alpha}^{\varepsilon} = \partial_{\alpha}, \partial_3^{\varepsilon} = \varepsilon^{-1} \partial_3 \tag{4.4}$$

4.2 The scaling of the variational problem

Inserting the upper scalings in the variational problem lead to the following:

Proposition 4.1. *The variational problem* (VP^{ε}) *is equivalent to the following scaled variational problem* ($SVP(\varepsilon)$)*: Find* $(u(\varepsilon)(t), \sigma(\varepsilon)) \in \vec{K}(\Omega) \times \mathbb{L}^2_s(\Omega), t \geq 0$ *such that*

$$\frac{\partial^2}{\partial t^2}\rho \int_\Omega u_3(\varepsilon)\, v_3 dx + \varepsilon^2 \frac{\partial^2}{\partial t^2}\rho \int_\Omega u_\alpha(\varepsilon)\, v_\alpha dx + \int_\Omega \sigma_{ij}(\varepsilon)\, \partial_j v_i dx$$

$$= L(v) + \langle \sigma_{33}(\varepsilon), \bar{v}_3 \rangle \; \forall v \in \vec{V}(\Omega), t > 0 \quad (4.5)$$

$$\langle \sigma_{33}(\varepsilon), \bar{v}_3 - \bar{u}_3(\varepsilon) \rangle \geq 0, \; \forall v_3 \in K(\Omega), t > 0 \quad (4.6)$$

$$u(\varepsilon)(.,0) = p(\varepsilon), \dot{u}(\varepsilon)(.,0) = q(\varepsilon) \quad (4.7)$$

$$\begin{cases} \sigma_{\alpha\beta}(\varepsilon) = \lambda e_{\gamma\gamma}(u(\varepsilon))\delta_{\alpha\beta} + 2\mu e_{\alpha\beta}(u(\varepsilon)) + \varepsilon^{-2}\lambda e_{33}(u(\varepsilon)) \\ \sigma_{\alpha 3}(\varepsilon) = \varepsilon^{-2} 2\mu e_{\alpha 3}(u(\varepsilon)) \\ \sigma_{33}(\varepsilon) = \varepsilon^{-4}\;(\lambda + 2\mu)\, e_{33}(u(\varepsilon)) + \varepsilon^{-2}\lambda e_{\gamma\gamma}(u(\varepsilon)) \end{cases} \quad (4.8)$$

where

$$e_{ij}(u(\varepsilon)) = \frac{1}{2}(\partial_i u_j(\varepsilon) + \partial_j u_i(\varepsilon))$$

$$L(v) = \int_\Omega f_i v_i dx + \int_{\Gamma_-} g_i \underline{v}_i d\Gamma$$

Since the problem (VP^{ε}) has at least a solution u^{ε} then the problem ($SVP(\varepsilon)$) has at least a solution $u(\varepsilon)$ with the regularity $u(\varepsilon) \in L^\infty(0, T, \vec{K}(\Omega))$, $\dot{u}(\varepsilon) \in L^\infty(0, T, (L^2(\Omega))^3)$ and $\ddot{u}(\varepsilon) \in \mathcal{D}'(0, T, (L^2(\Omega))^3)$. The tensor $\sigma(\varepsilon)$ belongs to $\mathcal{D}'(0, T, E_{ad}(g)) \cap L^\infty(0, T, (L^2(\Omega))^9)$ with $E_{ad}(g) = \{\tau \in \mathbb{L}^2_s(\Omega);\; \mathrm{div}\tau \in (L^2(\Omega))^3, \tau_{\alpha 3} = 0$ and $\tau_{33} \leq 0$ on $\Gamma_+; \tau n = g$ on $\Gamma_-\}$.

4.3 Two-dimensional problem

We assume that the scaled displacement-stress $(u(\varepsilon), \sigma(\varepsilon))$ admits a formal asymptotic expansion of the form:

$$(u(\varepsilon), \sigma(\varepsilon)) = (u^0, \sigma^0) + \varepsilon(u^1, \sigma^1) + \varepsilon^2(u^2, \sigma^2) + ..., \quad (4.9)$$

$$u^0 \in \vec{V}(\Omega), u^q \in (H^1(\Omega))^3, \sigma^0, \sigma^q \in \mathbb{L}^2_s(\Omega), q \in \{1, 2, \ldots\}, t > 0 \quad (4.10)$$

We introduce the space of Kirchhoff-Love admissible displacement

$$\begin{aligned} V_{KL}(\Omega) &= \{v = (v_i) \in \left(H^1(\Omega)\right)^3, e_{i3}(v) = 0\} \quad (4.11) \\ &= \left\{ \begin{array}{c} v = (v_i) \in \left(H^1(\Omega)\right)^3 \;/\; v_\alpha = \eta_\alpha - x_3\partial_\alpha\eta_3, v_3 = \eta_3 \text{ such that} \\ \eta_\alpha \in H^1_0(\omega), \eta_3 \in H^2_0(\omega) \end{array} \right\} \end{aligned}$$

Suppose that there exists $p^k, q^k, k = 0, 1, 2, ...$ such that $p(\varepsilon) = p^0 + \varepsilon p^1 + \varepsilon^2 p^2 + ...,\;\; q(\varepsilon) = q^0 + \varepsilon q^1 + \varepsilon^2 q^2 + ...$ with $p(\varepsilon) \to p(0)$ in $\vec{V}(\Omega)$ and $q_i(\varepsilon) \to q_i(0)$ in $L^2(\Omega)$ when ε goes to 0 . Suppose also that there exist φ_i, ψ_i independent of x_3 such that $p^0_\alpha = \varphi_\alpha - x_3\partial_\alpha\varphi_3, p^0_3 = \varphi_3$, and $q^0_\alpha = \psi_\alpha - x_3\partial_\alpha\psi_3, q^0_3 = \psi_3$.

In the next proposition, we give the problem that characterizes (u^0, σ^0) the leading term in the expansion of the scaled displacement-stress $(u(\varepsilon), \sigma(\varepsilon))$.

Proposition 4.2. *The leading term* (u^0, σ^0) *of the expansion* (4.9) *is a solution of the problem* (VP^0_{KL})*:*

$$\text{Find} \quad (u^0(t), \sigma^0(t)) \in \left(V_{KL}(\Omega) \cap \vec{K}(\Omega)\right) \times \mathbb{L}^2_s(\Omega),\ t \geq 0 \quad \text{such that}$$

$$\frac{\partial^2}{\partial t^2}\rho \int_\Omega u^0_3 v_3 dx + \int_\Omega \sigma^0_{ij}\partial_j v_i dx = L(v) + \left\langle \sigma^0_{33}, \bar{v}_3 \right\rangle, \forall v \in \vec{V}(\Omega),\ t > 0 \tag{4.12}$$

$$\left\langle \sigma^0_{33}, \bar{v}_3 - \bar{u}^0_3 \right\rangle \geq 0, \forall v_3 \in K(\Omega),\ \ t > 0 \tag{4.13}$$

$$\sigma^0_{\alpha\beta} = \lambda^* e_{\gamma\gamma}(u^0)\delta_{\alpha\beta} + 2\mu e_{\alpha\beta}(u^0) \ \text{ with } \lambda^* = \frac{2\lambda\mu}{\lambda + 2\mu} \tag{4.14}$$

$$u^0(.,0) = p^0, \dot{u}^0(.,0) = q^0 \tag{4.15}$$

where

$$e_{\alpha\beta}(u^0) = \frac{1}{2}(\partial_\alpha u^0_\beta + \partial_\beta u^0_\alpha).$$

Proof. The inverted constitutive equation reads $e^\varepsilon_{ij}(u^\varepsilon) = \lambda_1 \sigma^\varepsilon_{pp}\delta_{ij} + \frac{1}{2\mu}\sigma^\varepsilon_{ij}$ with $\lambda_1 = -\frac{\lambda}{2\mu(3\lambda + 2\mu)}$. Then the scaled deformation tensor reads

$$\begin{cases} e_{\alpha\beta}(u(\varepsilon)) = \lambda_1 \sigma_{\gamma\gamma}(\varepsilon)\delta_{\alpha\beta} + \dfrac{1}{2\mu}\sigma_{\alpha\beta}(\varepsilon) + \varepsilon^2 \lambda_1 \sigma_{33}(\varepsilon)\delta_{\alpha\beta}, \\ e_{\alpha 3}(u(\varepsilon)) = \varepsilon^2 \dfrac{1}{2\mu}\sigma_{\alpha 3}(\varepsilon), \\ e_{33}(u(\varepsilon)) = \varepsilon^2 \lambda_1 \sigma_{\gamma\gamma}(\varepsilon) + \varepsilon^4 \dfrac{\lambda + \mu}{\mu(3\lambda + 2\mu)}\sigma_{33}(\varepsilon). \end{cases} \tag{4.16}$$

Inserting (4.9) in the system (4.16), we obtain

$$e_{\alpha\beta}(u^0) = -\lambda_1 \sigma^0_{\gamma\gamma}\delta_{\alpha\beta} + \frac{1}{2\mu}\sigma^0_{\alpha\beta}, \tag{4.17}$$

$$e_{\alpha 3}(u^0) = 0, \tag{4.18}$$

$$e_{33}(u^0) = 0. \tag{4.19}$$

From (4.18) – (4.19) and the contact condition, we deduce that $u^0 \in V_{KL}(\Omega) \cap \vec{K}(\Omega))$. The equation (4.12) is obtained by inserting (4.9) in (4.5). By the same mean, we obtain (4.6). ■

In the next proposition, we re-write the problem (VP^0_{KL}) in terms of ξ_α and ξ_3. Hence we get a two dimensional problem $(P^b(0))$ whose solutions are ξ_α and ξ_3. The vector field (ξ_i) represents the (scaled) displacement of the middle surface ω of the plate.

Proposition 4.3. *If* (u^0, σ^0) *is a solution of* (VP^0_{KL}) *such that* $u^0_\alpha = \xi_\alpha - x_3\partial_\alpha\xi_3$ *and* $u^0_3 = \xi_3$*, with* ξ_α, ξ_3 *sufficiently smooth. Then* ξ_α, ξ_3 *verify with* σ^0_{33}*, at least formally, the two-dimensional boundary value problem* $(P^b(0))$*:*

$$\text{Find} \ \ \xi_\alpha \in H^1(\omega), \xi_3 \in H^2_0(\omega), \xi_3 \leq 0, \text{ for } \ \text{a.e } t \geq 0 \ \ \text{such that}$$

$$2\frac{\partial^2}{\partial t^2}\rho\xi_3 + \ k\Delta^2\xi_3 = h^1_1 + h^1_2 + h^0_3 + \sigma^0_{33} \ \text{ on } \omega \times]0, +\infty[\tag{4.20}$$

$$-\partial_\beta n_{\alpha\beta} = h^0_\alpha \quad \text{on} \ \ \omega \times]0, +\infty[\tag{4.21}$$

$$\sigma^0_{33}\xi_3 = 0 \ \text{ in } \omega \times]0, +\infty[, \sigma^0_{33} \leq 0 \ \ \text{in} \ \ H^{-2}(\omega) \tag{4.22}$$

$$\xi_i(.,0) = \varphi_i, \frac{\partial \xi_i}{\partial t}(.,0) = \psi_i \tag{4.23}$$

where

$$k = \frac{8}{3}\mu\frac{\lambda + \mu}{\lambda + 2\mu},\ \ h^0_i = \int_{-1}^{+1} f_i dx_3 + g^-_i,\ \ h^1_i = \int_{-1}^{+1} x_3\partial_i f_i dx_3 - \partial_i g^-_i, g^-_i = g_i(x_1, x_2, -1)$$

$$n_{\alpha\beta} = \frac{4\lambda\mu}{\lambda + 2\mu}e_{\gamma\gamma}(\xi)\delta_{\alpha\beta} + 4\mu e_{\alpha\beta}(\xi)$$

Proof. Let u^0 be a solution of the problem (VP^0_{KL}) then ξ_α, ξ_3 verify $\xi_\alpha \in H^1_0(\omega), \xi_3 \in H^2_0(\omega)$, $\xi_3 \leq 0$. Substituting

$$e_{\alpha\beta}(u^0) = e_{\alpha\beta}(\xi) - x_3 \partial_{\alpha\beta}\xi_3$$

in (4.14), we obtain

$$\sigma^0_{\alpha\beta} = \frac{1}{2} n_{\alpha\beta} + \frac{3}{2} x_3 m_{\alpha\beta} \tag{4.24}$$

with

$$\begin{aligned} n_{\alpha\beta} &= \frac{4\lambda\mu}{\lambda + 2\mu} e_{\gamma\gamma}(\xi)\,\delta_{\alpha\beta} + 4\mu e_{\alpha\beta}(\xi) \\ m_{\alpha\beta} &= -\frac{4}{3}\left(\frac{\lambda\mu}{\lambda+2\mu}\Delta\xi_3\delta_{\alpha\beta} + \mu\partial_{\alpha\beta}\xi_3\right) \end{aligned} \tag{4.25}$$

We take $v = (-x_3\partial_1\eta_3, -x_3\partial_2\eta_3, \eta_3)$ then the second term in the left hand side of the equilibrium equation in the problem (VP^0_{KL}) becomes

$$\begin{aligned} \int_\Omega \sigma^0_{\alpha\beta}\partial_\beta v_\alpha dx &= \int_\Omega -\frac{1}{2}\, n_{\alpha\beta} x_3 \partial_{\alpha\beta}\eta_3 dx + \int_\Omega -\frac{3}{2} x_3^2 m_{\alpha\beta}\partial_{\alpha\beta}\eta_3 dx \\ &= \int_\Omega x_3^2 \left(\frac{2\lambda\mu}{\lambda+2\mu}\Delta\xi_3\Delta\eta_3 + 2\mu\partial_{\alpha\beta}\xi_3\partial_{\alpha\beta}\eta_3\right) dx \\ &= \frac{4}{3}\int_\omega \left(\frac{2\lambda\mu}{\lambda+2\mu}\Delta\xi_3\Delta\eta_3 + 2\mu\partial_{\alpha\beta}\xi_3\partial_{\alpha\beta}\eta_3\right) dx'. \end{aligned}$$

For $\eta_3 \in \mathcal{D}(\Omega)$ we have,

$$\begin{aligned} \int_\omega \Delta\xi_3\Delta\eta_3 dx' &= \int_\omega \Delta^2\xi_3\eta_3 dx' \\ \int_\omega \partial_{\alpha\beta}\xi_3\partial_{\alpha\beta}\eta_3 dx' &= \int_\omega \Delta^2\xi_3\eta_3 dx' \end{aligned}$$

which, by density, still holds for elements of $H^2_0(\omega)$. Hence

$$\int_\Omega \sigma^0_{\alpha\beta}\partial_\beta v_\alpha dx = \frac{8}{3}\mu\frac{\lambda+\mu}{\lambda+2\mu}\int_\omega \Delta^2\xi_3\eta_3 dx'. \tag{4.26}$$

On the other hand, the right hand side of the equation becomes

$$\begin{aligned} L(v) + \langle\sigma^0_{33}, \bar{v}_3\rangle &= \int_\Omega f_\alpha v_\alpha dx + \int_\Omega f_3 v_3 dx + \int_{\Gamma_-} g^-_\alpha \underline{v}_\alpha d\Gamma + \int_{\Gamma_-} g^-_3 \underline{v}_3 d\Gamma \\ &+ \int_{\Gamma_+} \sigma^0_{33}\bar{v}_3 d\Gamma \\ &= \int_\omega \left\{\int_{-1}^{+1} x_3\partial_\alpha f_\alpha dx_3 - \partial_\alpha g^-_\alpha\right\}\eta_3 dx' \\ &+ \int_\omega \left\{\int_{-1}^{+1} f_3 dx_3 + g^-_3\right\}\eta_3 dx' + \int_\omega \sigma^0_{33}\eta_3 dx' \\ &= \int_\omega \left(h^0_3 + h^1_\alpha + \sigma^0_{33}\right)\eta_3 dx' \end{aligned} \tag{4.27}$$

with $h^0_i = \int_{-1}^{+1} f_i dx_3 + g^-_i$, $h^1_i = \int_{-1}^{+1} x_3\partial_i f_i dx_3 - \partial_i g^-_i$, g^-_i trace of g_i on Γ_-. Then, from (4.26) and (4.27) we obtain (4.20), in sense of distributions. By taking $\bar{v}_3 = 0$ (resp. $\bar{v}_3 = 2u^0_3$) in the inequality in the problem (VP^0_{KL}) we find $\langle\sigma^0_{33}, \xi_3\rangle = 0$ (resp. $\langle\sigma^0_{33}, \eta_3\rangle \geq 0$, for all $\eta_3 \in H^2_0(\omega)$ with $\eta_3 \leq 0$, which leads to $\sigma^0_{33} \leq 0$ in $H^{-2}(\omega)$). This proves (4.22). Now, we take $v = (\eta_1, \eta_2, 0)$ in the left hand side of the equilibrium equation in the problem (VP^0_{KL}), we obtain

$$\int_\Omega \sigma^0_{\alpha\beta}\partial_\beta \eta_\alpha dx = \int_\Omega f_\alpha \eta_\alpha dx + \int_{\Gamma_-} g^-_\alpha \eta_\alpha d\Gamma, \forall \eta_1, \eta_2 \in H^1_0(\omega).$$

Therefore

$$\int_\omega n_{\alpha\beta}\partial_\beta \eta_\alpha dx' = \int_\omega h^0_\alpha \eta_\alpha dx', \forall \eta_1, \eta_2 \in H^1_0(\omega),$$

hence we find (4.21) in sense of distributions. ∎

The next proposition is devoted to the characterization of σ^0_{i3}.

Proposition 4.4. *If (u^0, σ^0) is a solution of the problem (VP^0_{KL}) and ξ_α, ξ_3 verify the problem $(P^b(0))$, then u^0 and σ^0 are given by:*

$$u^0_\alpha = \xi_\alpha - x_3\partial_\alpha \xi_3, \; u^0_3 = \xi_3, \tag{4.28}$$

$$\sigma^0_{\alpha\beta} = \frac{1}{2}\, n_{\alpha\beta} + \frac{3}{2} x_3 m_{\alpha\beta}, \tag{4.29}$$

$$\sigma^0_{\alpha 3} = \frac{3}{4}(1 - x_3^2)\partial_\beta m_{\alpha\beta} + \frac{1}{2}(1 + x_3)\int_{-1}^{+1} f_\alpha dy_3 - \int_{-1}^{x_3} f_\alpha dy_3 + \frac{1}{2}(x_3 - 1) g^-_\alpha. \tag{4.30}$$

Finally σ^0_{33} satisfies in sense of distributions:

$$\partial_3\sigma^0_{33} = \rho\ddot{\xi}_3 - \partial_\alpha \sigma^0_{\alpha 3} - f_3 \text{ in } \Omega\times]0, +\infty[\,, \tag{4.31}$$

$$\sigma^0_{33} = -g^-_3 \text{ on } \Gamma_- \times]0, +\infty[\,, \tag{4.32}$$

$$\sigma^0_{33}\xi_3 = 0, \; \sigma^0_{33} \leq 0 \text{ on } \Gamma_+ \times]0, +\infty[\,. \tag{4.33}$$

Proof. The proof of this proposition follows essentially the same pattern as in [14, Thm 4.8-1, p.301] or in [13, Thm 1.7-1, p.38]. The equations (4.28) – (4.29) are already obtained in the Proposition 4.3. Using test function $v = (v_1, v_2, 0)$ in the equation (4.12), we obtain, *formally* that $\sigma^0_{\alpha 3}$ verifies in sense of distributions the quasistatic boundary value problem

$$\begin{cases} \partial_3\sigma^0_{\alpha 3} = -\partial_\beta\sigma^0_{\alpha\beta} - f_\alpha \text{ in } \Omega\times]0, +\infty[\\ \sigma^0_{\alpha 3} = -g^-_\alpha \text{ on } \Gamma_- \times]0, +\infty[\\ \sigma^0_{\alpha 3} = 0 \text{ on } \Gamma_+ \times]0, +\infty[\end{cases}$$

Integrating the first equation over $[-1, x_3]$ and taking in account the boundary conditions, we obtain (4.30).

For the computation of σ^0_{33}, we take as test functions $v = (0, 0, v_3)$ in the equation (4.12) then we obtain

$$\frac{\partial^2}{\partial t^2}\rho\int_\Omega u^0_3 v_3 dx + \int_\Omega \sigma^0_{33}\partial_3 v_3 dx = -\int_\Omega \sigma^0_{\alpha 3}\partial_\alpha v_3 dx + \int_\Omega f_3 v_3 dx + \int_{\Gamma_-} g_3 \underline{v}_3 d\Gamma + \int_{\Gamma_+} \sigma^0_{33}\bar{v}_3 d\Gamma, \forall v_3 \in V(\Omega), t > 0. \tag{4.34}$$

On the other hand, by Green's formula and taking into account the cancelations on the boundary Γ_0, we have *formally* that

$$\int_\Omega \sigma^0_{33}\partial_3 v_3 dx = -\int_\Omega \partial_3\sigma^0_{33} v_3 dx + \int_{\Gamma_+} \sigma^0_{33}\bar{v}_3 d\Gamma - \int_{\Gamma_-} \sigma^0_{33}\underline{v}_3 d\Gamma, \forall v_3 \in V(\Omega), t > 0 \tag{4.35}$$

and

$$\int_\Omega \sigma^0_{\alpha 3}\partial_\alpha v_3 dx = -\int_\Omega \partial_\alpha \sigma^0_{\alpha 3} v_3 dx, \forall v_3 \in V(\Omega), t > 0. \tag{4.36}$$

Inserting (4.35) and (4.36) in (4.34) we find that

$$\frac{\partial^2}{\partial t^2}\rho \int_\Omega u^0_3 v_3 dx + \int_\Omega \partial_3 \sigma^0_{33} v_3 dx = -\int_\Omega \partial_\alpha \sigma^0_{\alpha 3} v_3 dx - \int_\Omega f_3 v_3 dx - \int_{\Gamma_-} g_3 \underline{v}_3 d\Gamma - \int_{\Gamma_-} \sigma^0_{33}\underline{v}_3 d\Gamma, \forall v_3 \in V(\Omega)$$

We derive from the inequality (4.13) that σ^0_{33} must *formally* verifies (4.33). In summary the boundary value problem (4.31)-(4.33) is *formally* satisfied. ■

5 Convergence study

In this section, we suppose that $u(\varepsilon)$ is a solution of $(P(\varepsilon).V)$, the forces verify $f, \dot{f} \in L^\infty(0, T, L^2(\Omega))$ and $g, \dot{g} \in L^\infty(0, T, L^2(\Gamma_-))$ and we study the limit of the sequence $(u(\varepsilon))$, after that, we compare the results. To this end, we introduce the tensor $\kappa(\varepsilon, v)$ defined as following

$$\kappa_{\alpha\beta}(\varepsilon, v) = e_{\alpha\beta}(v), \kappa_{\alpha 3}(\varepsilon, v) = \varepsilon^{-1} e_{\alpha 3}(v), \kappa_{33}(\varepsilon, v) = \varepsilon^{-2} e_{33}(v) \tag{5.1}$$

Then, the bilinear form $a^\varepsilon(u(\varepsilon), v)$ can be rewritten as following

$$a^\varepsilon(u(\varepsilon), v) = \int_\Omega [\lambda \kappa_{ii}(\varepsilon, u)\kappa_{jj}(\varepsilon, v) + 2\mu \kappa_{ij}(\varepsilon, u)\kappa_{ij}(\varepsilon, v)] dx \tag{5.2}$$

The space $\mathbb{L}^2_s(\Omega)$ of symmetric, square integrable tensors equipped with the scalar product $\langle \sigma, \tau \rangle = \int_\Omega \sigma_{ij}\tau_{ij} dx$ is a real Hilbert space. As it is known, the norms

$$\|\sigma\|_A = \langle A\sigma, \sigma \rangle^{1/2}, \|\sigma\|_{0,\Omega} \tag{5.3}$$

are equivalent where $\langle A\sigma, \tau \rangle = \int_\Omega (\lambda \sigma_{ii}\tau_{jj} + 2\mu \sigma_{ij}\tau_{ij} dx)$.

Proposition 5.1. *If $u(\varepsilon)$ is a solution of the problem $(P(\varepsilon).V)$ then, for ε small enough, the sequences $(u(\varepsilon)), (\kappa_{ij}(\varepsilon))$ are bounded, respectively, in $L^\infty(0, T, \vec{V}(\Omega))$ and $L^\infty(0, T, \mathbb{L}^2_s(\Omega))$ hence, there exists a subsequences also denoted by $(u(\varepsilon))$ and $(\kappa_{ij}(\varepsilon))$ which admit a weak star limits respectively in $L^\infty(0, T, \vec{V}(\Omega))$ and $L^\infty(0, T, \mathbb{L}^2_s(\Omega))$ denoted respectively by $(u(0))$ and $(\kappa_{ij}(0))$.*

Proof. Taking $v = \dot{u}$ in the equation (4.5), we obtain

$$\rho \int_\Omega \ddot{u}_3(\varepsilon)\dot{u}_3(\varepsilon) dx + \varepsilon^2 \rho \int_\Omega \ddot{u}_\alpha(\varepsilon)\dot{u}_\alpha(\varepsilon) dx + a^\varepsilon(u(\varepsilon), \dot{u}(\varepsilon)) = L(\dot{u}(\varepsilon)) + \langle \sigma_{33}(\varepsilon), \dot{u}_3(\varepsilon) \rangle \tag{5.4}$$

By putting $v = u(t \pm h)$ in (4.6) and letting $h \to 0$ we obtain $\langle \sigma_{33}, \dot{u}_3 \rangle = 0$, then the equation (5.4) leads to the inequality

$$\rho \int_\Omega \ddot{u}_3(\varepsilon)\dot{u}_3(\varepsilon) dx + \varepsilon^2 \rho \int_\Omega \ddot{u}_\alpha(\varepsilon)\dot{u}_\alpha(\varepsilon) dx + a^\varepsilon(u(\varepsilon), \dot{u}(\varepsilon)) = L(\dot{u}(\varepsilon)) \tag{5.5}$$

We can rewrite this inequality as

$$\frac{d}{2dt}\left[\rho \int_\Omega |\dot{u}_3(\varepsilon)|^2 dx + \varepsilon^2 \rho \sum_{\alpha=1}^{2} \int_\Omega |\dot{u}_\alpha(\varepsilon)|^2 dx + a^\varepsilon(u(\varepsilon), u(\varepsilon))\right] = L(\dot{u}(\varepsilon)) \tag{5.6}$$

Integrating (5.6) from 0 to t on τ and taking in account the initial conditions, we get

$$\rho \int_\Omega |\dot{u}_3(\varepsilon)|^2 dx + \rho \sum_{\alpha=1}^{2} \int_\Omega |\varepsilon \dot{u}_\alpha(\varepsilon)|^2 dx + a^\varepsilon(u(\varepsilon), u(\varepsilon)) = 2\int_0^t L(\dot{u}(\tau))\, d\tau + \rho \int_\Omega |q_3|^2 dx + \rho \sum_{\alpha=1}^{2} \int_\Omega |\varepsilon q_\alpha|^2 dx + a^\varepsilon(p,p) \tag{5.7}$$

We have

$$\int_0^t L(\dot{u}(\tau))\, d\tau = \int_0^t \int_\Omega f \dot{u} dx d\tau + \int_0^t \int_{\Gamma_-} g \dot{u} d\Gamma d\tau \tag{5.8}$$

Using integration by parts, the inequality $ab \le \frac{\theta}{2}a^2 + \frac{1}{2\theta}b^2, \theta > 0$ and the assumptions on $u(x,0), \dot{u}(x,0)$, and on f, g which allow us to define f and g at $t=0$ (see [4] page 45), we obtain

$$\begin{aligned}
\int_0^t \int_\Omega f \dot{u} dx d\tau &= \int_\Omega (f(x,t)u(x,t) - f(x,0)u(x,0))dx - \int_0^t \int_\Omega \dot{f} u dx d\tau & (5.9)\\
&\le \frac{\theta_1}{2}||f||^2_{L^2(\Omega)} + \frac{1}{2\theta_1}||u||^2_{L^2(\Omega)} + \int_\Omega |f(x,0)u(x,0)|dx + \int_0^t \int_\Omega |\dot{f} u| dx d\tau & (5.10)\\
&\le c_1(\theta_1, T) + \frac{1}{2\theta_1}||u||^2_{L^2(\Omega)} + \frac{1}{2}\int_0^t \int_\Omega |u|^2 dx d\tau, t \in]0,T] & (5.11)
\end{aligned}$$

Using the same argumentation, and the continuity of trace mapping $H^1(\Omega) \to H^{1/2}(\Gamma_+)$ we obtain

$$\int_0^t \int_{\Gamma_-} g \dot{u} d\Gamma d\tau \le c_2(\theta_2, T) + \frac{c_3(T)}{2\theta_2}||u||^2_{1,\Omega}, t \in]0,T], \forall \theta_2 > 0, \tag{5.12}$$

Then there exist constants c_5, c_6 and c_7 (depending on T) such that

$$\int_0^t L(\dot{u}(\tau))\, d\tau \le c_5 + \frac{c_6}{\theta}\|u(\varepsilon)\|^2_{1,\Omega} + c_7 \int_0^t \int_\Omega |u(\varepsilon)|^2 dx d\tau \forall \theta > 0. \tag{5.13}$$

The assumptions on the limits of $p(\varepsilon)$ and $q(\varepsilon)$ yields the existence of $\varepsilon_0 > 0$ such that these limits are bounded independently of $\varepsilon \le \varepsilon_0$. Under adequate choice of θ and the coercivity of $a^\varepsilon(.,.)$ there exists constants c_8 and c_9 such that the inequality (5.8) becomes

$$\int_\Omega |\dot{u}_3(\varepsilon)|^2 dx + \sum_{\alpha=1}^{2} \int_\Omega |\varepsilon \dot{u}_\alpha(\varepsilon)|^2 dx + \|u(\varepsilon)\|^2_{1,\Omega} \le c_8 + c_9 \int_0^t \int_\Omega |u(\varepsilon)|^2 dx d\tau \tag{5.14}$$

Using Gronwall's lemma, we get

$$u(\varepsilon) \in L^\infty(0,T,\vec{V}(\Omega)) \text{ and } \dot{u}_3(\varepsilon), \varepsilon\dot{u}_\alpha(\varepsilon) \in L^\infty(0,T,L^2(\Omega)) \tag{5.15}$$

Using the continuity of the form $a^\varepsilon(.,.)$, the assumptions (5.1) and the equivalence of the norms (5.3), we get

$$\|e_{\alpha\beta}(u(\varepsilon))\|_{0,\Omega} \le c, \|e_{\alpha 3}(u(\varepsilon))\|_{0,\Omega} \le \varepsilon c, \|e_{33}(u(\varepsilon))\|_{0,\Omega} \le \varepsilon^2 c, \text{ for } a.e\ t \in]0,T] \tag{5.16}$$

■

Proposition 5.2. *The weak star limits $u(0)$ and $\kappa(0)$ verify the following*

$$u(0) \in V_{KL}(\Omega) \cap \vec{K}(\Omega), \text{ for } a.e\ t \in]0,T], \tag{5.17}$$

$$\kappa_{\alpha\beta}(0) = e_{\alpha\beta}(u(0)), \kappa_{\alpha 3}(0) = 0, \kappa_{33}(0) = \frac{-\lambda}{\lambda + 2\mu} e_{\alpha\alpha}(u(0)), \text{ for } a.e\ t \in]0,T]. \tag{5.18}$$

Proof. From (5.16) and the lower semi-continuity of the norm

$$\|e_{ij}(u(0))\|_{0,\Omega} \leq \liminf_{\varepsilon>0} \|e_{ij}(u(\varepsilon))\|_{0,\Omega}, \text{for } a.e\ t \in]0,T] \tag{5.19}$$

we deduce that $e_{33}(0) = e_{\alpha 3}(0) = 0$, for $a.e\ t \in]0,T]$.
From the weak closeness of $\vec{K}(\Omega)$, we deduce that

$$u(0) \in \vec{K}(\Omega), \text{ for } a.e\ t \in]0,T]$$

Then

$$u(0) \in L^\infty(0,T,V_{KL}(\Omega) \cap \vec{K}(\Omega)) \tag{5.20}$$

The equality $\kappa_{\alpha\beta}(0) = e_{\alpha\beta}(u(0))$ derives from $\kappa_{\alpha\beta}(u(\varepsilon)) \rightharpoonup \kappa_{\alpha\beta}(u(0))$ in $L^\infty(0,T,L^2(\Omega))$ weak star and $e_{\alpha\beta}(u(\varepsilon)) \rightharpoonup e_{\alpha\beta}(u(0))$ in $L^\infty(0,T,L^2(\Omega))$ and the uniqueness of the weak limit.
To prove $\kappa_{\alpha 3}(0) = 0\ a.e\ t \in]0,T]$, we choose as test function $\phi(t)(\varepsilon v_1, \varepsilon v_2, 0)$, $v_\alpha \in V(\Omega)$ and $\phi(t) \in \mathcal{D}(0,T)$ in (4.5) after that we integrate from 0 to T we obtain

$$\rho \int_0^T \phi(t) \int_\Omega \varepsilon^2 \ddot{u}_\alpha \varepsilon v_\alpha dx dt + \int_0^T \phi(t) a^\varepsilon(u(\varepsilon), \varepsilon v) dt = \int_0^T \phi(t) L(\varepsilon v) dt. \tag{5.21}$$

The term $\rho \int_0^T \phi(t) \int_\Omega \varepsilon^2 \ddot{u}_\alpha \varepsilon v_\alpha dx dt$ goes to 0 when $\varepsilon \to 0$. Indeed, integrating by parts we obtain $\rho \int_0^T \dot{\phi}(t) \int_\Omega \varepsilon^2 \dot{u}_\alpha \varepsilon v_\alpha dx dt \to 0$ when $\varepsilon \to 0$, reminding that $(\dot{\phi}(t) v_\alpha \in L^1(0,T,L^2(\Omega)))$.
The term

$$\begin{aligned}
\int_0^T \phi(t) a^\varepsilon(u(\varepsilon), \varepsilon v) dt &= \varepsilon \int_0^T \phi(t) \int_\Omega \lambda \kappa_{ii}(\varepsilon,u) \kappa_{jj}(\varepsilon,v) + 2\mu \kappa_{ij}(\varepsilon,u) \kappa_{ij}(\varepsilon,v) dx dt \\
&= \varepsilon \int_0^T \phi(t) \int_\Omega \lambda \kappa_{ii}(\varepsilon,u) e_{\beta\beta}(v) + 2\mu \kappa_{\alpha\beta}(\varepsilon,u) e_{\alpha\beta}(v) dx dt \\
&+ \int_0^T \phi(t) \int_\Omega 2\mu \kappa_{\alpha 3}(\varepsilon,u)(\partial_\alpha v_3 + \partial_3 v_\alpha) dx dt \\
&+ \frac{1}{\varepsilon} \int_0^T \phi(t) \int_\Omega (\lambda \kappa_{ii}(\varepsilon,u) + 2\mu \kappa_{33}(\varepsilon,u)) e_{33}(v) dx dt \\
&= \varepsilon \int_0^T \phi(t) \int_\Omega \lambda \kappa_{ii}(\varepsilon,u) e_{\beta\beta}(v) + 2\mu \kappa_{\alpha\beta}(\varepsilon,u) e_{\alpha\beta}(v) dx dt \\
&+ \int_0^T \phi(t) \int_\Omega 2\mu \kappa_{\alpha 3}(\varepsilon,u) \partial_3 v_\alpha dx dt,
\end{aligned} \tag{5.22}$$

due to the choice of the test function. We now pass to the limit in (5.21) and get

$$\lim_{\varepsilon \to 0} \int_0^T \phi(t) \int_\Omega 2\mu \kappa_{\alpha 3}(\varepsilon,u) \partial_3 v_\alpha dx dt = 0, \forall \phi \in \mathcal{D}(0,T)$$

Then

$$\int_\Omega 2\mu \kappa_{\alpha 3}(u(0)) \partial_3 v_\alpha dx = 0\ a.e\ for\ t \in]0,T], \forall v_\alpha \in V(\Omega).$$

Recalling that of [15, Theorem 3.4-1] states that any $w \in L^2(\Omega)$ such that $\int_\Omega w \partial_3 v dx = 0$ for all $v \in \mathcal{C}^\infty(\bar{\Omega})$ that satisfy $v = 0$ on $\gamma \times [-1,1]$ then $w = 0$, we deduce that $\kappa_{\alpha 3}(u(0)) = 0$ in Ω in sense of distribution for a.e $t \in]0,T]$. To prove $\kappa_{33}(0) = \frac{-\lambda}{\lambda+2\mu} e_{\alpha\alpha}(u(0))$ we use as test function $\phi(t)(0,0,\varepsilon^2 v_3)$, $\bar{v}_3 = 0$, $v_3 \in V(\Omega)$ and $\phi(t) \in \mathcal{D}(0,T)$ in (4.5) after that we integrate from 0 to T in (4.5) then, we pass to the limit and obtain

$$\int_\Omega (\lambda \kappa_{ii}(0) + 2\mu \kappa_{33}(0)) \partial_3 v_3 dx = 0\ a.e\ for\ t \in]0,T], \forall v_3 \in V(\Omega).$$

The result follows. ■

Proposition 5.3. *The sequence* $(\sigma_{33}(\varepsilon))$ *verifies*

$$\sigma_{33}(\varepsilon) \rightharpoonup \sigma_{33}(0) \text{ in } L^\infty(0,T,H^{-2}(\omega)) \text{ weak star} \tag{5.23}$$

and verifies with $u(0)$ *the inequality*

$$\langle\sigma_{33}(0), u_3(0)\rangle \leq \lim_{\varepsilon\to 0}\langle\sigma_{33}(\varepsilon), u_3(\varepsilon)\rangle \text{ in } L^\infty(0,T). \tag{5.24}$$

Proof. We recall that among unknowns of the problem is the contact stress $\sigma_{33}(\varepsilon)$ on Γ_+. Now, we prove that also this quantity has weak limit when ε tends to zero.
We prove now that

$$\int_\Omega \ddot{u}_3(\varepsilon)v_3dx \quad\to\quad \int_\Omega \ddot{u}_3(0)v_3dx \text{ in } \mathcal{D}'(0,T) \tag{5.25}$$

Let $\phi \in \mathcal{D}(0,T)$, we have

$$\int_0^T \phi(t)\int_\Omega \ddot{u}_3(\varepsilon)v_3dx = -\int_0^T\int_\Omega \dot{u}_3(\varepsilon)\dot{\phi}(t)v_3dx.$$

From (5.15) we deduce that there exists $\chi \in L^\infty(0,T,L^2(\Omega))$ such that

$$\dot{u}_3(\varepsilon) \rightharpoonup \chi \text{ in } L^\infty(0,T,L^2(\Omega)) \text{ weak star} \tag{5.26}$$

i.e

$$\int_0^T(\dot{u}_3(\varepsilon), g(t))dt \to \int_0^T(\chi, g(t))dt, \forall g \in L^1(0,T,L^2(\Omega)). \tag{5.27}$$

For $g(x,t) = \phi(t)w(x)$ such that $\phi(t) \in \mathcal{D}(0,T)$ and $w \in L^2(\Omega)$, we have

$$\textstyle\int_0^T(\dot{u}_3(\varepsilon), \phi(t)w(\cdot))dt \quad=\quad -\int_0^T(u_3(\varepsilon), \dot{\phi}(t)w(\cdot))dt. \tag{5.28}$$

On the other hand $u(\varepsilon)$ converges strongly to $u(0)$ in the space $L^\infty(0,T,L^2(\Omega))$. Therefore

$$\int_0^T(\dot{u}_3(\varepsilon), \phi(t)w(x))dt \to -\int_0^T(u_3(0), \dot{\phi}(t)w(\cdot))dt = \int_0^T(\dot{u}_3(0), \phi(t)w(\cdot))dt.$$

From the density of functions $\phi(t)w(x)$ in $L^1(0,T,L^2(\Omega))$ we deduce that

$$\int_0^T(\dot{u}_3(\varepsilon), g(t))dt \to \int_0^T(\chi, g(t))dt, \forall g \in L^1(0,T,(L^2(\Omega))) \tag{5.29}$$

with $\chi = \dot{u}_3(0)$. That means

$$\dot{u}_3(\varepsilon) \rightharpoonup \dot{u}_3(0) \text{ in } L^\infty(0,T,L^2(\Omega)) \text{ weak star} \tag{5.30}$$

Therefore $\chi = \dot{u}_3(0)$ by virtue of the uniqueness of weak limit. Since $\dot{u}_3(\varepsilon) \rightharpoonup \dot{u}_3(0)$ in $L^\infty(0,T,L^2(\Omega))$ weak star and $\dot{\phi}v_3 \in L^1(0,T,L^2(\Omega))$ then

$$\textstyle\int_0^T\int_\Omega \dot{u}_3(\varepsilon)\dot{\phi}(t)v_3dx \quad\to\quad \int_0^T\int_\Omega \dot{u}_3(0)\dot{\phi}(t)v_3dx = -\int_0^T\int_\Omega \ddot{u}_3(0)\phi(t)v_3dx \tag{5.31}$$

From (4.5), we get for all $v \in V_{KL}(\Omega)$ and $\phi \in \mathcal{D}(0,T)$

$$\begin{aligned}\int_0^T \phi(t)\left<\sigma_{33}(\varepsilon), v_3\right> dt &= -\int_0^T \phi(t)L(v)dt + \rho\int_0^T \phi(t)\left<\ddot{u}_3(\varepsilon), v_3\right> dt \\ &+ \varepsilon^2\rho\int_0^T \phi(t)\left<\ddot{u}_\alpha(\varepsilon), v_\alpha\right> dt + \int_0^T \phi(t)a^\varepsilon(u(\varepsilon), v)dt\end{aligned} \tag{5.32}$$

when ε goes to 0, we obtain that for all $v \in V_{KL}(\Omega)$ and $\phi \in \mathcal{D}(0,T)$

$$\int_0^T \phi(t)\left<\sigma_{33}(\varepsilon), v_3\right> dt \to \int_0^T \phi(t)(a_*^0(u(0), v) - L(v) + \rho\left<\ddot{u}_3(0), v_3\right>)dt$$

Then

$$\left<\sigma_{33}(\varepsilon), v_3\right> \to a_*^0(u(0), v) - L(v) + \rho\left<\ddot{u}_3(0), v_3\right> \text{ in } \mathcal{D}'(0,T)$$

such that

$$a_*^0(u(0), v) = \int_\Omega \sigma_{\alpha\beta}(0)\partial_\beta v_\alpha dx \text{ with } \sigma_{\alpha\beta}(0) = \lambda^* e_{\gamma\gamma}(u(0))\delta_{\alpha\beta} + 2\mu e_{\alpha\beta}(u(0)) \text{ with } \lambda^* = \frac{2\lambda\mu}{\lambda + 2\mu}$$

The map $v_3 \longrightarrow a_*^0(u(0), v) - L(v) + \rho\left<\ddot{u}_3(0), v_3\right>$ defines a linear form on $H_0^2(\omega)$. Then the sequence $\sigma_{33}(\varepsilon)$ admits a weak star limit in $L^\infty(0,T,H^{-2}(\omega))$ denoted by $\sigma_{33}(0)$ which verifies the equation

$$\rho\left<\ddot{u}_3(0), v_3\right> + a_*^0(u(0), v) = L(v) + \left<\sigma_{33}(0), v_3\right>, \forall v \in V_{KL}(\Omega), \ a.e\ t \in]0,T[\tag{5.33}$$

From (5.33), we deduce $\sigma_{33}(0)$ verifies with $u(0)$ the equation

$$\rho\left<\ddot{u}_3(0), u(0)\right> + a_*^0(u(0), u(0)) = L(u(0)) + \left<\sigma_{33}(0), u_3(0)\right> \ a.e\ t \in]0,T[. \tag{5.34}$$

We prove now that

$$\left<\sigma_{33}(0), v_3 - u_3(0)\right> \geq 0, \ \forall v_3 \in K(\Omega), \ a.e\ t \in]0,T[. \tag{5.35}$$

To do this, we must verify that

$$\left<\sigma_{33}(0), u_3(0)\right> \leq \lim_{\varepsilon \longrightarrow 0}\left<\sigma_{33}(\varepsilon), u_3(\varepsilon)\right>, \text{ for } \ a.e\ t \in]0,T[. \tag{5.36}$$

then letting ε goes to 0, we obtain (5.35).
We take in (4.6) first $v_3 = 0$ and secondly $v_3 = 2u_3$ and obtain that $\left<\sigma_{33}(\varepsilon), u_3(\varepsilon)\right> = 0$ then

$$\left<\sigma_{33}(\varepsilon), u_3(\varepsilon)\right> \rightharpoonup 0 \ in\ L^\infty(0,T) \text{ weak star}. \tag{5.37}$$

We have that

$$\|\kappa(0)\|_A^2 = a_*^0(u(0), u(0))$$

hence for a.e $t \in]0,T[$

$$\begin{aligned}\left<\sigma_{33}(0), u_3(0)\right> &= \|\kappa(0)\|_A^2 - L(0) + \rho\left<\ddot{u}_3(0), u_3(0)\right> \\ &\leq \liminf_{\varepsilon>0}\left(\|K(\varepsilon)\|_A^2 - L(\varepsilon) + \rho\left<\ddot{u}_3(\varepsilon), u_3(\varepsilon)\right>\right) \\ &\leq \liminf_{\varepsilon>0}\left<\sigma_{33}(\varepsilon), u_3(\varepsilon)\right> \\ &\leq \lim_{\varepsilon\longrightarrow 0}\left<\sigma_{33}(\varepsilon), u_3(\varepsilon)\right>\end{aligned}$$

Indeed, since $u_3(\varepsilon)$ converges weak star to $u_3(0)$ in $H_0^2(\Omega)$ for a.e $t \in]0,T[$ then converges strongly in $H_0^1(\Omega)$ and $\ddot{u}_3(\varepsilon)$ converges weak star to $\ddot{u}_3(0)$ in $H^{-1}(\Omega)$ for a.e $t \in]0,T[$ then, we have proved (5.36) therefore (5.35). For the initial conditions, $u(0)$ verifies

$$u_{|_{t=0}}(0) = p(0), \dot{u}_{|_{t=0}}(0) = q(0)$$

■

In summary we have proved the following theorem:

Theorem 5.1. *If $u(\varepsilon)$ is a solution of the problem $(P(\varepsilon).V)$ then*

$$\begin{aligned} u(\varepsilon) &\rightharpoonup u(0) \text{ in } L^\infty(0,T,\vec{V}(\Omega)) \ weak-* \\ \sigma_{33}(\varepsilon) &\rightharpoonup \sigma_{33}(0) \text{ in } L^\infty(0,T,H^{-2}(\omega)) \ weak-* \end{aligned}$$

where $u(0)$ is a solution of the problem $(VP_{KL}(0))$:

$$\begin{gathered} Find \ \ u(0) \in \ V_{KL}(\Omega) \cap \vec{K}(\Omega) \, for \ a.e \ t \in [0,T] \, such \ that \\ \rho \langle \ddot{u}_3(0), v_3 \rangle + a_*^0(u(0), v) = L(v) + \langle \sigma_{33}(0), v_3 \rangle \, , \forall v \in V_{KL}(\Omega) \\ \langle \sigma_{33}(0), v_3 - u_3(0) \rangle \geq 0, \ \forall v_3 \in K(\Omega) \\ u_{|t=0}(0) = p(0), \dot{u}_{|t=0}(0) = q(0) \end{gathered}$$

with

$$a_*^0(u(0),v) = \int_\Omega \sigma_{\alpha\beta}(0) \partial_\beta v_\alpha dx, \sigma_{\alpha\beta}(0) = \lambda^* e_{\gamma\gamma}(u(0)) \delta_{\alpha\beta} + 2\mu e_{\alpha\beta}(u(0)), \ \lambda^* = \frac{2\mu\lambda}{\lambda + 2\mu}.$$

Notice that the solution (u^0, σ^0) of problem (VP^0_{KL}) from Proposition 4.2 is also a solution of problem $(VP_{KL}(0))$, but since the uniqueness of the solution of this last problem is not garanteed we cannot conclude that $(u^0, \sigma^0) = (u(0), \sigma(0))$.

6 Conclusion

We have shown that the weak limit of the subsequence $(u(\varepsilon), \sigma_{33}(\varepsilon))$ of the three-dimensional unilateral problem is solution of a two-dimensional unilateral problem. Furthermore the solution of the other two-dimensional problem which derives from the asymptotic method is also solution of the one deduced from the convergence method. As there is no uniqueness result for this problem, we cannot conclude that the solutions coincide. Nevertheless is it possible to prove the strong convergence of the whole sequence by using the variational inequality (3.1)? Another open problem is to prove similar results for the time-dependent Signorini problem with Coulomb friction.

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