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THÈME

ETUDE DE QUELQUES PROBLEMES ELLIPTIQUES NON LINEAIRES

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To my Mother,

my wife

and all my Familly.

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Preface

Many physical, chemical and biological phenomena are described by nonlinear partial differential equations, however; many of them do not, in general, possess smooth solutions. It is, therfore, essential to find another kind of appropriate solutions. Namely, the notion of weak solutions.

The aim of this work is to discuss some nonlinear elliptic problems in bounded domains with smooth boundaries and apply the maximum principal to their solutions. To do so, we need to introduce some theoretical notions of partial differential equations and recall the main properties of Sobolev spaces which are powerful tools to study these equations.

The first chapter is devoted to a short description of the physical and chemical aspect of Laplace's and Poisson's equations, classification of PDE of second order and the different types of boundary conditions. This chapter ends by a biological example which explains how to obtain a PDE from the data.

In chapter 2, we introduce the definition of weak solution of elliptic problems and the relationship between classical and variational formulation of PDE's.

A large part of this chapter is devoted to the definition and some important properties of $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$.

The third chapter is devoted to illustrate the techniques used in the study of linear PDE's by applying them to a various elliptic problems. The last problem is an excellent example where we applied simultaneously the Lax-Milgram lemma and the trace theorem.

In [7], Chipot studied the following problem

$$\begin{cases} -\sum\limits_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a(x,u)\frac{\partial u}{\partial x_i}\right) = f \quad \text{in } \Omega \\ u = 0 \quad \text{in } \partial\Omega \end{cases}.$$

By using the compacity method, he proved that, for every $f \in L^2(\Omega)$, this problem has a weak solution

$$u \in H_0^1(\Omega).$$

In chapter4, we gave a detailed proof to the problem discussed by Chipot then we extended his result to problem

$$\begin{cases} -\sum\limits_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(a_{ij}(x,u) \frac{\partial u}{\partial x_i} \right) = f \quad \text{in } \Omega \\ u = 0 \quad \text{in } \partial\Omega \end{cases}.$$

Chapter 5 is devoted to the study of some problems involving the $p-$Laplace operator. Precisely, we studied the problem

$$\begin{cases} -\sum\limits_{i=1}^{N} \frac{\partial}{\partial x_i} \left(a(x,u) \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) = f \quad \text{in } \Omega \\ u = 0 \quad \text{in } \partial\Omega \end{cases},$$

which generalizes a similar problem studied by Lions in [17], namely

$$\begin{cases} -\sum\limits_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) = f \quad \text{in } \Omega \\ u = 0 \quad \text{in } \partial\Omega \end{cases}.$$

He proved, using the monotonicity method, that the problem has a weak solution $u \in W_0^{1,p}(\Omega)$ for every $f \in W^{-1,p'}(\Omega)$.

In the 6^{th} chapter we study the problem

$$\begin{cases} -\sum\limits_{i=1}^{N} \frac{\partial}{\partial x_i} \left(a(x,u) \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) + b(x)u = f \quad \text{in } \Omega \\ u = 0 \quad \text{in } \partial\Omega \end{cases},$$

where we need to apply some Sobolev compact embedding theorems.

In chapter 7 we apply the maximum principal to the solutions of the above problems, in particular the problems involving the $p-$Laplacian operator.

Contents

Chapter 1

Introduction

1.1 Elliptic partial differential equations origins

The study of partial differential equations started in the work of Euler, d'Alembert, Lagrange and Laplace as a central tool in the description of continum mechanics, and more generally, as the principal mode of analytical study of models in physical science [6].

Many physical processes are described by equations that involve physical quantities together with their partial derivatives. Among such processes are flow of liquids, deformation of solid bodies, chemical reactions, electromagnetic and many others [23].

In this section we discuss the physical aspects of some problems that will be used later as model problems. Among the most important of all partial differential equations are Laplace's equation

$$\Delta u = 0 \tag{1.1}$$

and Poisson's equation

$$-\Delta u = f$$

where, in both equations $u : \Omega \to \mathbb{R}$ is the unknown function defined in a domain $\Omega \subset \mathbb{R}^N$, $f : \Omega \to \mathbb{R}$ is a given function and Δ is the second-order operator defined by

$$\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$$

In a typical interpretation u denotes the density of some quantity (e.g. a chemical concentration) in equilibrium or the displacement of elastic membrane or electrostatic potential. Then if V is any subregion within Ω with a smooth boundary, the net flux of u through the boundary ∂V is zero. That is,

$$\int_{\partial V} F.\eta ds = 0,$$

where F denotes the flux density and η is the unit outer normal field.

In many instances it is physically reasonable to assume the flux F is proportional to the gradient of u, but points in the opposite direction $F = -a\nabla u$, where $a > 0$ is the constant of proportion.

Using the Green formula, we have

$$\int_{V} divF dx = \int_{\partial V} F.\eta ds = 0$$

and so

$$divF = 0 \text{ in } \Omega \tag{1.2}$$

since V is arbitrary.

Substituting for F into (1.2), we obtain

$$div\,(-a\nabla u) = 0,$$

thus,

$$\Delta u = 0,$$

which is the Laplace equation [14].

1.2 Partial differential equations classification

As the Laplace equation (1.1) is the prototype of elliptic equation, the heat equation

$$\frac{\partial u}{\partial t} - \Delta u = 0,$$

and the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0,$$

are respectively the prototypes of parabolic and hyperbolic partial differential equations, where the variable t describes the time.

In the middle of the second decade of the twentieth century, Hadamared proposed to find general classes which are a generalizations of the Laplace equation, the heat equation and the wave equation and having distinctive properties for their solutions in terms of characteristic polynomials. We, thus, obtain a basic class of second-order operator [6].

Consider a second-order partial differential equation in the form

$$-\sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x) \frac{\partial u}{\partial x_i} + a_0(x) u = f(x), \tag{1.3}$$

where a_{ij}, b_i, a_0 and f are continuous functions defined in a domain $\Omega \subset \mathbb{R}^N$. The principal part of the left hand side is

$$L_0(u) = \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}. \tag{1.4}$$

We can assume without loss generality that the matrix

$$A = (a_{ij}(x)),$$

is symmetric [23].

The differential operator (1.4), or the equation (1.3) is said to be elliptic at $x \in \Omega$, if the matrix $A(x)$ is positive definite, which means that all the eigenvalues of A are non-zero and have the same sign. The parabolic case is characterized by one zero eigenvalue with all other eigenvalues having the same sign. In the hyperbolic case, however, the matrix A is invertible but the sign of one eigenvalue is different from the signs of all the other eigenvalues [15].

An equation is called elliptic, parabolic, or hyperbolic in Ω if it is elliptic, parabolic, or hyperbolic everywhere in Ω, respectively [21].

This classification was subsequently extended to nonlinear partial differential equations, to linear PDE of arbitrary order, and to systems [6].

In particular, for second-order partial differential equations of general form

$$F\left(x,u,p_i,p_{ij}\right)=0, \tag{1.5}$$

where $p_i = \frac{\partial u}{\partial x_i}$ and $p_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$, the equation (1.5) is called elliptic, parabolic, or hyperbolic in Ω, if the matrix

$$A\left(x\right)=\left(\frac{\partial F}{\partial p_{ij}}\left(x\right)\right),$$

has the same properties of A in the linear case, respectively [23].

1.3 Boundary conditions

In practical applications one does not usually want to solve problems posed in all the space; rather one wants to solve these problems on some domain, subject to certain conditions. Thus, we wish to impose additional conditions upon the solution u, typically prescribing the values of u or of certain first derivatives of u on the boundary of the domain or part of it. One knows, that there are very specific kinds of boundary conditions usually associated with each equations. Here are some one [19]:

1) *Dirichlet condition*: it specifies the values of the solution on the boundary of the domain. The question of finding solutions to the problem

$$\begin{cases} L\left(u\right)=f \text{ in } \Omega \\ u=g \text{ in } \partial\Omega \end{cases},$$

where L is a differential operator, is known as *Dirichlet problem.*

2) *Neumann condition:* it specifies the values of the derivative of a solution is on the boundary of the domain. The problem of finding a function satisfying

$$\begin{cases} L\left(u\right)=f \text{ in } \Omega \\ \frac{\partial u}{\partial \eta}=g \text{ in } \partial\Omega \end{cases},$$

is known as *Neumann problem.* Here, f, g are given functions defined in Ω and $\partial\Omega$ respectively, and η the unit outer normal field to the boundary $\partial\Omega$.

3) *Robin condition*: it is a specification of a linear combination of the values of a function

and the values of its derivative on the boundary of the domain. We suppose that the unknown function u satisfies, in addition to the partial differential equation, the condition

$$\alpha u + \beta \frac{\partial u}{\partial \eta} = g \ \text{ in } \partial\Omega,$$

where α and β are some non-zero constants.

4) *Cauchy condition*: specifies both the values that to take a solution and its normal derivative on the boundary of the domain. It corresponds to imposing both a Dirichlet and a Neumann boundary condition,

$$\begin{cases} L(u) = f \text{ in } \Omega \\ u = g \text{ in } \partial\Omega \\ \frac{\partial u}{\partial \eta} = h \text{ in } \partial\Omega \end{cases}$$

where f is given in Ω and g, h are given in $\partial\Omega$.

Well-posed problems

We say that a problem is well-posed (in the sense of Hadamard) if there exists a solution, the solution is unique and depends continuously on the data, if these conditions do not hold a problem is said to be ill-posed [21].

The third requirement is important, because in applications, the boundary data are obtained through measurements and thus are given only up to certain error margins, and small measurement errors should not change the solution drastically [16].

A chemical aspect of second-order elliptic equation

The second order-elliptic partial differential equation

$$-\sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x) \frac{\partial u}{\partial x_i} + c(x) u = f$$

generalizes Laplace's and Poisson's equations. As in the derivation of Laplace's equation set forth, u represents, for instance, the chemical concentration at equilibrium within a region Ω, the second-order term $\sum_{i,j=1}^{N} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}$ represents the diffusion of u within Ω, the coefficients a_{ij} describe the anisotropic, heterogenous nature of the medium. The first order term $\sum_{i=1}^{N} b_i \frac{\partial u}{\partial x_i}$ represents transport within Ω and the term cu describes the local creation or depletion of the chemical.

1.4 A problem in biology

As an example, let us consider a biological problem [7].

Let Ω be a bounded domain of $\mathbb{R}^3$. Suppose that Ω is a Petri box filled with some nutrient and a colony of bacteria. We denote by $u(x_1, x_2, x_3)$ the density of bacteria at the point (x_1, x_2, x_3). In the microworld Ω there is three aspects of the life: birth, death and motion. That is to say, in Ω some new bacteria are coming to life, some other are dying and some others are moving from one place to another.

We will consider that these three phenomena balance each other in such a way that the density of bacteria remains unchanged with time.

Let us analyze the phenomenon of diffusion. We note by $u(x)$ the density of bacteria in x. Then, the diffusion velocity v, of migration at a point x, in the direction η, is given by

$$v(\eta) = -a(x)\left.\frac{d}{dt}u(x + t\eta)\right|_{t=0}\eta,$$

which we can write also

$$v(\eta) = -a(x)(\nabla u(x).\eta)\eta,$$

where $a(x)$ is the coefficient of proportionality, which is a positive constant depending on x and η is the unit vector in $\mathbb{R}^3$. So, it is natural to assume that the average of the velocity v on S_2, is given by

$$\overrightarrow{v} = \frac{1}{|S_2|}\int_{S_2} v(\eta)\, d\sigma(\eta),$$

where S_2 is the unit sphere in $\mathbb{R}^3$, $|S_2|$ its area and $d\sigma(\eta)$ denotes the element of surface area on S_2.

The i^{th} entry of the vector v is

$$v_i = \frac{1}{|S_2|}\int_{S_2} -a(x)(\nabla u(x).\eta)\eta_i d\sigma(\eta).$$

For obvious symmetry reasons, we have

$$\int_{S_2} \eta_i^2 d\sigma(\eta) = \frac{1}{3} \int_{S_2} \left(\eta_1^2 + \eta_2^2 + \eta_3^2\right) d\sigma(\eta) = \frac{|S_2|}{3}$$

and

$$\int_{S_2} \eta_i \eta_j d\sigma(\eta) = 0, \ \forall i \neq j.$$

So, for $i = 1, 2, 3$, we obtain

$$v_i = -\frac{a}{3} \frac{\partial u}{\partial x_i}$$

Thus, replacing $\frac{a}{3}$ by a

$$v = -a \nabla u.$$

Consider an elementary volume V included in Ω with outward unit normal $\overrightarrow{n}$.

The flux of bacteria through the boundary ∂V of V is given by

$$\int_{\partial V} \overrightarrow{v} . \overrightarrow{n} d\sigma(x) = \int_{\partial V} -a \left(\nabla u . \overrightarrow{n}\right) d\sigma(x),$$

where $d\sigma$ denotes the element of surface area on ∂V.

The death in Ω occurs at a rate proportional to the density of population through a factor λ. So, in V we observe the disappearance of the quantity

$$\lambda \int_V u dx,$$

where $dx = dx_1 dx_2 dx_3$ denotes the volume measure in $\mathbb{R}^3$.

If we denote by f the density of bacteria supplied from outside, then there appears in V a quantity

$$\int_V f dx.$$

So, clearly in order for the density u remain constant in time, we must have a balance of population

$$\int_{\partial V} -a \left(\nabla u . \overrightarrow{n}\right) d\sigma + \lambda \int_V u dx = \int_V f dx. \tag{1.6}$$

Assuming u smooth we have by the divergence theorem

$$\int_{\partial V} a\,(\nabla u.\overrightarrow{n})\,d\sigma = \int_V div\,(a\nabla u)\,dx.$$

So (1.6) can now be written

$$\int_V (-div\,(a\nabla u) + \lambda u)\,dx = \int_V f dx,$$

and this for any volume V, this implies that

$$-div\,(a\nabla u) + \lambda u = f, \text{ in } \Omega.$$

Since the density of bacteria has to vanish on the boundary $\partial\Omega$ of Ω, the problem to solve is to find u which satisfies

$$\begin{cases} -div\,(a\nabla u) + \lambda u = f, \text{ in } \Omega \\ u = 0, \text{ in } \partial\Omega \end{cases}$$

which is a Dirichlet problem.

For more examples see [9], [10] and [13].

Chapter 2

Sobolev Spaces

2.1 Introduction

An important systematic machinery to carry through the study of PDE was introduced by S. L. Sobolev in the mid of 1930's: the definition of new classes of function spaces, named Sobolev spaces.

Together with the L^p spaces, Sobolev spaces became one of the most powerful tools in analysis, they are indispensable for a theoretical analysis of partial differential equations, as well as being necessary for the analysis of some numerical methods for solving such equations [6].

2.2 Motivation

Assume that Ω is a bounded domain with a Lipschitz boundary and consider the elliptic equation

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(a_{ii}(x) \frac{\partial u}{\partial x_i} \right) + au = f, \tag{2.1}$$

where $a_{ii} \in C^1(\Omega)$, a and f belong to $C(\Omega)$.

Suppose that u is a classical solution to the problem (2.1) with the homogeneous Dirichlet boundary conditions

$$u(x) = 0, \forall x \in \partial\Omega. \tag{2.2}$$

In the classical treatment of second-order partial differential equations, the solution and its derivatives up to order two are required to be continuous functions, then, u mast be assumed to belong to $C^2(\Omega) \cap C^1(\overline{\Omega})$, satisfy (2.1) everywhere in Ω and vanishes in the boundary $\partial\Omega$.

However, the requirement made on a_{ii}, a and f do not guarantee the existence of a solution to the problem (2.1), (2.2), with the strong regularity that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$.

Weak formulation:

In order to reduce the strong regularity assumed on the classical solution, we multiply both sides of (2.1) by a function $\varphi \in C_0^1(\Omega)$ and integrate over Ω,

$$-\int_\Omega \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(a_{ii}\frac{\partial u}{\partial x_i}\right)\varphi dx + \int_\Omega au\varphi dx = \int_\Omega f\varphi dx$$

thus, using the Green's formula, the first term in the left-hand side becomes

$$-\sum_{i=1}^N \int_\Omega \varphi \frac{\partial}{\partial x_i}\left(a_{ii}\frac{\partial u}{\partial x_i}\right) dx = -\int_{\partial\Omega} \varphi \left(\sum_{i=1}^N a_{ii}\frac{\partial u}{\partial x_i}\eta_i\right) d\sigma + \sum_{i=1}^N \int_\Omega a_{ii}\frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_i} dx, \qquad (2.3)$$

where $\eta = (\eta_1, \eta_2, ..., \eta_N)$ is the unit outer normal field and $d\sigma$ is an elementary surface in $\partial\Omega$.

Since φ vanishes in $\partial\Omega$, the first term in the right-hand side of (2.3) vanishes and we have

$$\begin{aligned} \int_\Omega \left(\sum_{i=1}^N a_{ii}\frac{\partial u}{\partial x_i}.\frac{\partial \varphi}{\partial x_i} + au\varphi\right) dx &= \int_\Omega f\varphi dx \qquad (2.4) \\ \int_\Omega \left(\sum_{i=1}^N a_{ii}\frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_i} + au\varphi - f\varphi\right) dx &= 0, \end{aligned}$$

Thus, the identity (2.4) was derived under very strong regularity assumptions $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and $\varphi \in C_0^1(\Omega)$, but all integrals in (2.4) remain finite when these assumptions are weakened to a_{ii}, $a \in L^\infty(\Omega)$, u, $\frac{\partial u}{\partial x_i}$, $f \in L^p(\Omega)$ and φ, $\frac{\partial \varphi}{\partial x_i} \in L^{p'}(\Omega)$.

Thus, we can change the given problem (2.1), (2.2) by the problem:

Given $f \in L^p(\Omega)$, find $u \in L^p(\Omega)$ such that $\frac{\partial u}{\partial x_i} \in L^p(\Omega)$, for all i, $1 \leq i \leq N$, and satisfies

$$\int_\Omega \left(\sum_{i=1}^N a_{ii}\frac{\partial u}{\partial x_i}.\frac{\partial \varphi}{\partial x_i} + au\varphi\right) dx = \int_\Omega f\varphi dx, \quad \forall \varphi \in C_0^1(\Omega).$$

Notice that the assumption $f \in L^p(\Omega)$ can be further weakened to an other assumption which we will mention later.

The last formulation of the problem (2.1), (2.2) is called the weak formulation or variational formulation of the given problem (2.1), (2.2).

So far we have said nothing about the existence and uniqueness of solutions in the variational formulation of the boundary value problem (2.1), (2.2), because to deal adequately with these topics it is necessary to work in the framework of new spaces called Sobolev spaces. In the following sections we introduce these spaces and recall some of their properties which we need to use later.

2.3 Weak derivative

Assume that $u \in C^1(\Omega)$, then, by integration by parts, we have

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} dx = -\int_\Omega \frac{\partial u}{\partial x_i} \varphi dx, \quad \forall \varphi \in C_0^1(\Omega) \text{ for } i = 1, 2, ..., N \tag{2.5}$$

where $C_0^1(\Omega)$ is the space of continuously differentiable functions with compact support in Ω, there are no boundary terms, since φ vanishes near $\partial\Omega$.

Notice that the left-hand side of (2.5) makes sense if u is only locally integrable i.e. $u \in L^1_{loc}(\Omega)$, then $\frac{\partial u}{\partial x_i}$ has no obvious meaning if u is not $C^1(\Omega)$ function.

Definition: Let $u \in L^1_{loc}(\Omega)$, if there exists a function $v_i \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} dx = -\int_\Omega v_i \varphi dx, \quad \forall \varphi \in C_0^1(\Omega) \text{ for } 1 \le i \le N,$$

we say that v_i is the i^{th} weak first partial derivative of u.

More generally, suppose that $u \in L^p(\Omega)$, since $C_0^1(\Omega)$ is dense in $L^{p'}(\Omega)$ for all p' such that $1 \le p' < +\infty$, then $u\frac{\partial \varphi}{\partial x_i} \in L^1(\Omega)$ and the integral

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} dx,$$

makes sense for all $\varphi \in C_0^1(\Omega)$.

Thus, we say that a function $u \in L^p(\Omega)$, has an ith-weak first partial derivative, if there exists a function $v_i \in L^r(\Omega)$ for $1 \leq r < +\infty$, such that

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} dx = -\int_\Omega v_i \varphi dx, \forall \varphi \in C_0^1(\Omega), \tag{2.6}$$

v_i is the ith-weak first partial derivative of u and denoted by $\frac{\partial u}{\partial x_i}$.

Using the fact that if $f \in L^1_{loc}(\Omega)$ satisfies

$$\int_\Omega f \varphi dx = 0, \quad \forall \varphi \in C_0(\Omega),$$

then,

$$f = 0 \text{ a.e. in } \Omega,$$

we can prove that the weak derivative is unique almost everywhere and if $u \in C^1(\Omega)$ the weak partial derivative coincide with the usual partial derivative [19].

More generally, let $m \geq 1$ be an integer number and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N) \in \mathbb{N}^N$ be a multi-index. We say that u has a weak partial derivative of order α, if there exists a function v_α such that

$$\int_\Omega u \frac{\partial^{|\alpha|} \varphi}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2 ... \partial^{\alpha_N} x_N} dx = (-1)^{|\alpha|} \int_\Omega v_\alpha \varphi dx, \quad \forall \varphi \in C_0^\infty(\Omega).$$

v_α is denoted by

$$\frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2 ... \partial^{\alpha_N} x_N} \text{ or } D^\alpha u.$$

2.4 Sobolev space

Let Ω be an open subset of $\mathbb{R}^N$ and $p \in [1, +\infty]$ a real number, the Sobolev space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega); \forall i = 1, 2, ..., N, \quad \exists v_i \in L^p(\Omega), \text{ such that } v_i = \frac{\partial u}{\partial x_i} \right\},$$

where $\frac{\partial u}{\partial x_i}$ is the weak derivative of u.

Similarly we define Sobolev spaces of higher order $W^{m,p}(\Omega)$, for a given integer $m \geq 1$ and a real $p \in [1, +\infty]$,

$$W^{m,p}(\Omega) = \left\{ u \in L^p(\Omega), \ \ \forall \alpha \in \mathbb{N}^N, |\alpha| \leq m, \ \ \exists v_\alpha \in L^p(\Omega), \text{ such that } v_\alpha = D^\alpha u \ \right\},$$

where the derivative $D^\alpha u$ must be understand in the weak sense.

We equip the space $W^{m,p}(\Omega)$, for $1 \leq p < +\infty$, by the norm

$$\|u\|_{W^{m,p}} = \|u\|_{L^p} + \sum_{1 \leq |\alpha| \leq m} \|D^\alpha u\|_{L^p},$$

or the equivalent norm

$$\begin{aligned} \|u\|_{W^{m,p}} &= \left(\int_\Omega |u|^p \, dx + \sum_{1 \leq |\alpha| \leq m} \int_\Omega |D^\alpha u| \, dx \right)^{\frac{1}{p}} \\ &= \left(\|u\|_{L^p}^p + \sum_{1 \leq |\alpha| \leq m} \|D^\alpha u\|_{L^p}^p \right)^{\frac{1}{p}}, \end{aligned}$$

$W^{m,\infty}(\Omega)$ is equipped by the norm

$$\|u\|_{W^{m,\infty}} = \max_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{L^\infty},$$

where $D^\alpha u = u$, when $|\alpha| = 0$.

For $p = 2$, $W^{m,p}(\Omega)$ is denote by $H^m(\Omega)$.

Remark: In the case where $\Omega = \mathbb{R}^N$, we can define $H^m(\mathbb{R}^N)$ to be the subspace of $L^2(\mathbb{R}^N)$ constituted by all functions $u \in L^2(\mathbb{R}^N)$ such that

$$\left(1 + |\xi|^2\right)^{\frac{m}{2}} \widehat{u} \in L^2(\mathbb{R}^N), \quad \xi \in \mathbb{R}^N,$$

where $\widehat{u}$ is the Fourier transformation of u. Also,

$$\|u\|_{H^m} = \left\| \left(1 + |\xi|^2\right)^{\frac{m}{2}} \widehat{u} \right\|_{L^2(\mathbb{R}^N)},$$

For the proof see [20].

2.5 Elementary properties

In this section we recall some results and theorems which we need to use later. For the proofs we refer the reader to mentioned references.

Proposition 1:

1) $W^{m,p}(\Omega)$ equipped with the norms defined above is a Banach space and $H^m(\Omega)$ equipped with the inner product

$$\langle u, v\rangle = \int_\Omega uv dx + \sum_{1\leq|\alpha|\leq m} \int_\Omega D^\alpha u . D^\alpha v dx$$

is a Hilbert space.

2) $W^{m,p}(\Omega)$ is separable space for $1 \leq p < +\infty$ and reflexive for $1 < p < +\infty$.

Proof: see [5] and [14].

Theorem 2 (Meyers and Serrin): Let

$$\begin{aligned} S &= \{u \in C^\infty(\Omega) \text{ such that } \|u\|_{W^{1,p}} < +\infty\} \\ &= C^\infty(\Omega) \cap W^{1,p}(\Omega), \end{aligned}$$

then S is dense in $W^{1,p}(\Omega)$.

Proof: See [1].

Proposition 3:

If Ω is bounded then $C^m\left(\overline{\Omega}\right) \subset W^{m,p}(\Omega)$.

Proof:

Since Ω is bounded then $\overline{\Omega}$ is compact.

Let $u \in C^m\left(\overline{\Omega}\right)$, then u, $D^\alpha u \in C\left(\overline{\Omega}\right)$ for every $|\alpha| \leq m$ and u, $D^\alpha u$ attained their maximums in $\overline{\Omega}$. Thus, there exists $M > 0$ such that

$$\|D^\alpha u\|_{C(\overline{\Omega})} = \max_{x\in\overline{\Omega}} |D^\alpha u(x)| \leq M, \ \forall |\alpha| \leq m,$$

then,

$$\int_\Omega |D^\alpha u|^p dx \leq M^p meas(\Omega) < +\infty.$$

Corollary 4: Assume that Ω is a Lipschitz domain, then for $1 \leq p < \infty$, $C^{\infty}\left(\overline{\Omega}\right)$ is dense in $W^{m,p}(\Omega)$.

For the proof see [4].

Definition:

An open subset $\Omega \subset \mathbb{R}^N$ is said to be of class C^1 if for each $x \in \partial\Omega$ there exists a neighborhood U of x in $\mathbb{R}^N$ and a bijection $H : Q \to U$ such that

$$\begin{aligned} H &\in C^1(\overline{Q}), \\ H^{-1} &\in C^1(\overline{U}), \\ H\left(Q_+\right) &= U \cap \Omega \end{aligned}$$

and

$$H\left(Q_0\right) = U \cap \partial\Omega,$$

where

$$\begin{aligned} Q &= \left\{x = \left(x', x_n\right); \quad \left|x'\right| < 1 \text{ and } \left|x_n\right| < 1\right\}, \\ Q_+ &= \left\{x \in Q; \quad x_n > 0\right\}, \\ Q_0 &= \left\{x \in Q; \quad x_n = 0\right\}. \end{aligned}$$

Theorem 5 (Extension theorem) :

Suppose that Ω is of class C^1 with bounded boundary $\partial\Omega$. Then there exists a linear extension operator

$$P : W^{1,p}\left(\Omega\right) \to W^{1,p}(\mathbb{R}^N),$$

such that, for all $u \in W^{1,p}\left(\Omega\right)$ we have

i) $Pu|_{\Omega} = u$

ii) $\left\|Pu\right\|_{L^p\left(\mathbb{R}^N\right)} \leq C \left\|u\right\|_{L^p(\Omega)}$

iii) $\left\|Pu\right\|_{W^{1,p}\left(\mathbb{R}^N\right)} \leq C \left\|u\right\|_{W^{1,p}(\Omega)}$.

where C depends only on p and Ω.

Corollary 6: Suppose that Ω is of class C^1 and let $u \in W^{1,p}\left(\Omega\right)$ be given with $1 \leq p < +\infty$.

Then there exists a sequence $\{u_k\} \subset C_0^\infty\left(\mathbb{R}^N\right)$ such that

$$u_k|_\Omega \to u \text{ in } W^{1,p}(\Omega).$$

Remark: The above corollary means that the restrictions of functions of $C_0^\infty\left(\mathbb{R}^N\right)$ in Ω is dense in $W^{1,p}(\Omega)$. Note that it is not true if Ω is not of class C^1.

2.6 Sobolev Inequalities

2.6.1 Continuous embedding :

Definition : Let X and Y be Banach spaces such that $X \subset Y$. We say that X is continuously embedded into Y, and write $X \hookrightarrow Y$, if the identity operator

$$I : X \to Y$$

is continuous, i.e. $\exists C > 0$ such that

$$\|x\|_Y \leq C \|x\|_X, \quad \forall x \in X.$$

Theorem 7 : (Gagliardo, Niremberg, Sobolev)

Let $1 \leq p < N$ and set

$$p^* = \frac{Np}{N-p},$$

then

$$W^{1,p}\left(\mathbb{R}^N\right) \hookrightarrow L^{p^*}(\mathbb{R}^N)$$

and there exists a constant C, depending on p and N only, such that

$$\|u\|_{L^{p^*}} \leq C \|\nabla u\|_{L^p}, \quad \forall u \in W^{1,p}(\mathbb{R}^N).$$

Theorem 8 : Let Ω be a bounded and open subset of $\mathbb{R}^N$ with C^1-boundary. Assume that $1 \leq p < N$, then

$$W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$$

and there exists a constant C, depending on p, N and Ω only, such that

$$\|u\|_{L^{p^*}} \leq C \|\nabla u\|_{W^{1,p}}, \quad \forall u \in W^{1,p}(\Omega).$$

Remark : For p and p^* defined above, by using the fact that, if $u \in L^p(U) \cap L^{p^*}(U)$ then

$$u \in L^q(U), \quad \forall q \in [p, p^*],$$

we can prove that

$$W^{1,p}(U) \subset L^q(U), \quad \forall q \in [p, p^*],$$

with continuous embedding, where $U = \mathbb{R}^N$ or $U = \Omega$ and Ω is bounded with C^1 bounded boundary.

Corollary 9 : $(p = N)$

$$W^{1,N}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N), \quad \forall q \in [N, +\infty[, \tag{2.7}$$

with continuous embedding [5].

Theorem 10 (Morrey) : Let $p > N$ then

$$W^{1,p}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N), \tag{2.8}$$

with continuous embedding [5].

Remark :

Assume that Ω is an open subset of C^1 bounded boundary, then we have the same results as in (2.7) and (2.8) with replacing $\mathbb{R}^N$ by Ω.

$$\begin{array}{lll} \text{for } p = N & \text{we have } W^{1,p}(\Omega) \subset L^q(\Omega), & \forall q \in [p, +\infty[, \\ \text{for } p > N & \text{we have } W^{1,p}(\Omega) \subset L^\infty(\Omega), & \end{array}$$

with continuous embedding [5].

2.6.2 Compact embedding

Definition: Let X and Y be two normed spaces such that $X \subset Y$. We say that X is compactly embedded in Y and write

$$X \hookrightarrow\hookrightarrow Y,$$

if

$i)$ There exists a constant C such that

$$\|x\|_Y \leq C \|x\|_X, \quad \forall x \in X.$$

$ii)$ Every bounded set in X is precompact in Y [4].

Theorem 11 (Rellich, Kondrachov): Suppose that Ω is bounded and of C^1-boundary. We have

$$\begin{array}{lll} \text{if } 1 \leq p < N & \text{then } W^{1,p}(\Omega) \subset L^q(\Omega), & \forall q \in [1, p^*[\,, \\ \text{if } p = N & \text{then } W^{1,p}(\Omega) \subset L^q(\Omega), & \forall q \in [1, +\infty[\,, \\ \text{if } p > N & \text{then } W^{1,p}(\Omega) \subset C(\overline{\Omega}), & \end{array}$$

with compact embedding [5].

Remark : Using the compact embedding of $W^{1,p}(\Omega)$ in $L^q(\Omega)$, we can extract from every bounded sequence $\{u_n\} \subset W^{1,p}(\Omega)$ a subsequence $\{u_{n_k}\}$ such that

$$u_{n_k} \to u \text{ in } L^q(\Omega),$$

where $u \in W^{1,p}(\Omega)$, also

$$u_{n_k} \rightharpoonup u \text{ in } W^{1,p}(\Omega)$$

and

$$u_{n_k} \to u \text{ a.e. in } \Omega.$$

Here, $u_{n_k} \rightharpoonup u$ means that $\{u_{n_k}\}$ is weakly converges to u, which means that for every f in the dual of $W^{1,p}(\Omega)$ we have

$$\langle f, u_{n_k} \rangle \to \langle f, u \rangle \text{ in } \mathbb{R}.$$

2.7 $W_0^{1,p}(\Omega)$ and its properties

Definition :

Let $1 \leq p < +\infty$. We denote by $W_0^{1,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$.

Thus, $u \in W_0^{1,p}(\Omega)$ if and only if there exists a sequence $\{u_k\} \subset C_0^\infty(\Omega)$, such that

$$u_k \to u \text{ in } W^{1,p}(\Omega).$$

Properties : 1) $W_0^{1,p}(\Omega)$ equipped with the norm induced by $W^{1,p}(\Omega)$ norm is a separable Banach space, it is reflexive for $1 < p < +\infty$.

2) $H_0^1(\Omega) = W_0^{1,2}(\Omega)$ is a Hilbert space with respect to the $H^1(\Omega)$ inner product.

Remarks : 1) Note that since $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$ then,

$$W_0^{1,p}(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N),$$

however, in general, for $\Omega \subsetneq \mathbb{R}^N$,

$$W_0^{1,p}(\Omega) \neq W^{1,p}(\Omega).$$

2) We can prove, by using a regularized sequence, that the closure of $C_0^1(\Omega)$ in $W^{1,p}(\Omega)$ is $W_0^{1,p}(\Omega)$.

2.7.1 Trace theorem

Notice that a function u of $W^{1,p}(\Omega)$ is only defined a.e. in Ω, so if also $u \in C(\overline{\Omega})$, then clearly u has usual values on $\partial\Omega$, but there are no meaning to the restriction of u at $\partial\Omega$, which is of negligible measure. The notion of trace operator resolves this problem.

Theorem 12: (**Trace theorem**):

Suppose that $1 \leq p < +\infty$ and assume that Ω is a bounded domain with C^1-boundary.

Then there exists a continuous linear mapping

$$\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

such that

i) If $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$, then

$$\gamma u = u|_{\partial\Omega}$$

and

ii) There exists a constant C depending only on p and Ω such that

$$\|\gamma u\|_{L^p(\partial\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}, \quad \forall u \in W^{1,p}(\Omega).$$

Proof : See [8], [7], [14] and [25].

Definition : We call γu the trace of u on $\partial\Omega$.

Theorem 13: (Trace of functions in $W_0^{1,p}(\Omega)$)

Let Ω be a bounded domain with boundary of class C^1. Suppose further that $u \in W^{1,p}(\Omega)$, then

$$u \in W_0^{1,p}(\Omega) \text{ if and only if } \gamma u = 0 \text{ on } \partial\Omega.$$

2.7.2 Poincaré's inequality

Suppose that Ω is bounded then for all p such that $1 \leq p \leq +\infty$ we have

$$\|u\|_{L^p} \leq C \|\nabla u\|_{L^p}, \quad \forall u \in W_0^{1,p}(\Omega),$$

where C is depending only on p and Ω.

Proof: See [24], [2].

Remarks :

1) Poincaré's inequality does not hold in $W_0^{1,p}(\Omega)$ if Ω contains arbitrarily large balls; i.e., if there exists a sequence $r_n \to \infty$ and points $x_n \in \Omega$ such that $B(x_n, r_n) \subset \Omega$.

2) If Ω is included in a strip of width d, i.e., there exists $\xi \in \mathbb{R}^N$ with $|\xi| = 1$ and

$\Omega \subset \left\{ x \in \mathbb{R}^N / \alpha < \xi.x < \beta \right\}$ and $d = \beta - \alpha$, then

$$\|u\|_{L^p} \leq C_0 \|\nabla u\|_{L^p}, \quad \forall u \in W_0^{1,p}(\Omega),$$

where C_0 is a universal constant; i.e., independent of which Ω.

3) If $p = \infty$ Poincaré's inequality holds on $W_0^{1,\infty}(\Omega)$ if and only if there exists $M < \infty$ such that

$$d(x, \partial\Omega) \leq M, \quad \forall x \in \Omega,$$

where $d(.,.)$ is the Euclidian distance.

Proof : See [24].

Corollary 14 : Suppose that Ω is bounded then $\|\nabla u\|_{L^p}$ is a norm on $W_0^{1,p}(\Omega)$, equivalent to the norm $\|u\|_{W^{1,p}(\Omega)}$.

Proof : We have

$$\|\nabla u\|_{L^p} \leq \|u\|_{W^{1,p}}, \quad \forall u \in W_0^{1,p}(\Omega).$$

By using Poincaré's inequality we have

$$\begin{aligned} \|u\|_{W^{1,p}} &= \|u\|_{L^p} + \|\nabla u\|_{L^p} \\ &\leq (C+1) \|\nabla u\|_{L^p}. \end{aligned}$$

Thus,

$$\|\nabla u\|_{L^p} \leq \|u\|_{W^{1,p}} \leq C' \|\nabla u\|_{L^p}.$$

Remark : The same result holds true if Ω has a finite width.

2.7.3 Dual of $W_0^{1,p}(\Omega)$

Definition : For $1 \leq p < \infty$ and its conjugate p', we denote by $W^{-1,p'}(\Omega)$, the dual space of $W_0^{1,p}(\Omega)$, in particular the dual of $H_0^1(\Omega)$, is denoted by $H^{-1}(\Omega)$.

Properties :

1) By identifying $L^2(\Omega)$ to its dual, we obtain

$$H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega),$$

where the embedding is continuous and dense.

2) Suppose that Ω is bounded then,

$$W_0^{1,p}(\Omega) \subset L^2(\Omega) \subset W^{-1,p'}(\Omega), \text{ if } \frac{2N}{N+2} \leq p < \infty,$$

where the embedding is continuous and dense.

3) If Ω is unbounded then,

$$W_0^{1,p}(\Omega) \subset L^2(\Omega) \subset W^{-1,p'}(\Omega), \text{ if } \frac{2N}{N+2} \leq p < 2.$$

Proposition 15 : $W^{-1,p'}(\Omega)$ is a subspace of $\mathfrak{D}'(\Omega)$ and it can be shown that the disrtibutions in $W^{-1,p'}(\Omega)$ are of the form

$$F = f_0 + \sum_{i=1}^{N} f_i,$$

where $f_i \in L^{p'}(\Omega)$ for $0 \leq i \leq N$.

Thus,

$$\langle F, \varphi \rangle = \int_\Omega f_0 \varphi + \sum_{i=1}^{N} \int_\Omega f_i \frac{\partial \varphi}{\partial x_i}, \forall \varphi \in W_0^{1,p}(\Omega)$$

with

$$\|F\|_{W^{-1,p'}} = \max_{0 \leq i \leq n} \|f_i\|_{L^{p'}}.$$

Moreover, if Ω is bounded we can take $f_0 = 0$.

Proof: See [5], [2] and [8].

Chapter 3

Some linear problems

The aim of this chapter is to familiarize ourselves with some elliptic problems by studying simple ones.

3.1 A homogeneous Dirichlet problem

Let us consider the problem:

Find u solution to

$$\begin{cases} -\Delta u + u = f \ \text{ in } \Omega \\ u = 0 \text{ in } \partial\Omega \end{cases} \tag{3.1}$$

where Ω is an open and bounded subset of $\mathbb{R}^N$ and f is a given function defined on Ω.

Recall that a classical solution to (3.1) is a function $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$, which verifies

$$-\Delta u + u = f \quad \text{in} \ \ \Omega$$

and vanishes on $\partial\Omega$ and a weak solution is a function $u \in H_0^1(\Omega)$ which verifies

$$\sum_{i=1}^{N} \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} + \int_{\Omega} uv = \int_{\Omega} fv, \quad \forall v \in H_0^1(\Omega). \tag{3.2}$$

Before studying this problem let us recall some topological concepts which serve us later.

3.2 Some topological concepts

Let H be a real Hilbert space, equipped with the inner product $\langle ., . \rangle$ and let $|.|_H$ be the associated norm.

3.2.1 Riesz-Fréchet theorem

Theorem 1 (Riesz-Fréchet representation theorem) : For any continuous linear functional φ on H there exists a unique $u \in H$ such that

$$\varphi(v) = \langle u, v \rangle, \quad \forall v \in H.$$

Moreover,

$$\|\varphi\|_{H'} = |u|_H .$$

Proof : If $\varphi = 0$ it is sufficient to take $u = 0$.

Suppose that $\varphi \neq 0$ and let $M = \varphi^{-1}(0)$, then M is closed and $M \neq H$. Thus, we can choose $w \in M^{\perp}$ such that $w \neq 0$.

Let $u = \frac{\varphi(w)}{|w|_H^2} w$, then $u \in M^{\perp}$ and for any $v \in H$,

$$\frac{\varphi(v)}{\varphi(w)} w \in M^{\perp}.$$

Also, there exists $z \in M$ such that

$$v = z + \frac{\varphi(v)}{\varphi(w)} w;$$

therefore,

$$v - \frac{\varphi(v)}{\varphi(w)} w \in M,$$

hence

$$\left\langle u, v - \frac{\varphi(v)}{\varphi(w)} w \right\rangle = 0,$$

then

$$\langle u, v \rangle = \frac{\varphi(v)}{\varphi(w)} \langle u, w \rangle$$

replacing u in the right-hand side we get

$$\begin{aligned} \langle u, v\rangle &= \frac{\varphi(v)}{\varphi(w)}\left\langle \frac{\varphi(w)}{|w|_H^2}w, w\right\rangle \\ &= \varphi(v). \end{aligned}$$

Moreover, by using Cauchy Shwartz inequality, we have

$$\begin{aligned} \|\varphi\| &= \sup_{|v|_H=1} |\langle u, v\rangle| \\ &\leq \sup_{|v|_H=1} |u|_H\, |v|_H \\ \|\varphi\| &\leq |u|_H\,, \end{aligned} \tag{3.3}$$

also,

$$\varphi\left(\frac{u}{|u|_H}\right) \leq \|\varphi\|\,,$$

and

$$\begin{aligned} \varphi\left(\frac{u}{|u|_H}\right) &= \left\langle u, \frac{u}{|u|_H}\right\rangle \\ &= |u|_H\,, \end{aligned}$$

then,

$$|u|_H \leq \|\varphi\|\,. \tag{3.4}$$

Thus, from (3.3) and (3.4) we have

$$\|\varphi\| = |u|_H\,.$$

To show the uniqueness of u, one simply notes that, if there exist $u, u' \in H$ such that

$$\varphi(v) = \langle u, v\rangle\,, \quad \forall v \in H$$

and

$$\varphi(v) = \langle u', v\rangle\,, \quad \forall v \in H$$

then,

$$\left\langle u-u', v\right\rangle = 0, \quad \forall v \in H.$$

By choosing $v = u - u'$ we get

$$\begin{aligned} \left\langle u-u', u-u'\right\rangle &= 0, \\ \left|u-u'\right|_H &= 0, \end{aligned}$$

therefore,

$$u = u'.$$

See also [18].

3.2.2 Lax Milgram lemma

Definition : A bilinear form $a : H \times H \to \mathbb{R}$ is said to be:

$i)$ continuous, if there exists a constant $C > 0$ such that

$$|a(u,v)| \leq C\,|u|_H\,|v|_H\,, \quad \forall u, v \in H,$$

$ii)$ coercive or $H-$elliptic, if there exists a constant $\alpha > 0$ such that

$$a(u,u) \geq \alpha\,|u|_H^2\,, \quad \forall u \in H.$$

Theorem 2 : Let $K \subset H$ be non empty, closed and convex subset. Then, for every $f \in H$, there exists a unique $u \in K$ such that

$$|f-u|_H = \min_{v \in K} |f-v|_H\,,$$

u is the orthogonal projection of f onto K. Moreover, u is characterized by

$$\begin{cases} u \in K \\ \langle f-u, v-u\rangle \leq 0, \quad \forall v \in K \end{cases}$$

and the map $P_K : H \to K$ defined by

$$P_K(f) = u$$

is a Lipschitz function with

$$|P_K(f_1) - P_K(f_2)| \leq |f_1 - f_2|, \quad \forall f_1, f_2 \in H, \tag{3.5}$$

see [5].

Theorem 3 (Stampacchia theorem): Let $a(.,.) : H \times H \to \mathbb{R}$ be a continuous bilinear and coercive form and $K \subset H$ be a non empty, closed and convex subset.

Then, for every $\varphi \in H'$ there exists a unique element $u \in K$ such that

$$a(u, v-u) \geq \varphi(v-u), \quad \forall v \in K.$$

If, in addition, $a(.,.)$ is symmetric, then u is characterized by

$$\begin{cases} u \in K \\ \frac{1}{2}a(u,u) - \varphi(u) = \min_{v \in K} \left\{ \frac{1}{2}a(v,v) - \varphi(v) \right\}. \end{cases}$$

Proof: By using Riesz-Fréchet theorem for φ, there exists a unique $f \in H$ such that

$$\varphi(v) = \langle f, v \rangle, \quad \forall v \in H.$$

Let $u \in K$ be the orthogonal projection of f onto K, then, from **Theorem 3**

$$\langle f - u, v - u \rangle \leq 0, \quad \forall v \in K \tag{3.6}$$

Also, for a fixed $w \in H$, the map

$$v \to a(w, v)$$

is a linear and continuous form on H, then by Riesz-Fréchet theorem, there exists a unique $w' \in H$ such that

$$a(w,v) = \left\langle w', v \right\rangle, \quad \forall v \in H.$$

Let the operator $A : H \to H$ defined by

$$Aw = w',$$

it is a linear continuous operator and satisfies

$$a(w,v) = \langle Aw, v \rangle, \quad \forall v \in H. \tag{3.7}$$

Indeed, it's easy to show that A is linear. To prove the continuity, we use the fact that

$$|a(w,v)| \leq C\,|w|_H\,|v|_H\,, \quad \forall w, v \in H$$

and (3.7) to obtain

$$|\langle Aw, v \rangle| \leq C\,|w|_H\,|v|_H\,, \quad \forall w, v \in H,$$

by replacing v by Aw in the last equality we arrive at

$$|Aw|^{\,2} \leq C\,|w|_H\,|Aw|_H\,, \quad \forall w \in H,$$

So, if $Aw \neq 0$, we easily get

$$|Aw| \leq C\,|w|_H\,, \quad \forall w \in H, \tag{3.8}$$

which still true even if $Aw = 0$. Therefore, A is continuous.

Moreover, from the coercivity property of $a(.,.)$ we have

$$|\langle Aw, w \rangle| \geq \alpha\,|w|_H^2\,, \quad \forall w \in H. \tag{3.9}$$

Let ρ be a positive constant, which will be fixed later, and define a map S by

$$\begin{aligned} S &: K \to K \\ S(w) &= P_K(\rho f - \rho Aw + w)\,. \end{aligned}$$

$S(w)$ is the orthogonal projection of $\rho f - \rho Aw + w$ onto K, then, from (3.6) we have

$$\langle \rho f - \rho Aw + w - S(w), v - S(w) \rangle \leq 0, \quad \forall v \in K. \tag{3.10}$$

Also, from (3.5) we have

$$\begin{aligned} |S(w_1) - S(w_2)| &\leq |(\rho f - \rho Aw_1 + w_1) - (\rho f - \rho Aw_2 + w_2)|, \quad \forall w_1, w_2 \in K, \\ &\leq |(w_1 - w_2) - \rho (Aw_1 - Aw_2)|, \quad \forall w_1, w_2 \in K, \end{aligned}$$

then,

$$|S(w_1) - S(w_2)|^2 \leq |w_1 - w_2|^2 + \rho^2 |Aw_1 - Aw_2|^2 - 2\rho \langle Aw_1 - Aw_2, w_1 - w_2 \rangle .$$

By inserting the inequalities (3.8) and (3.9) in the last inequality it becomes

$$\begin{aligned} |S(w_1) - S(w_2)|^2 &\leq |w_1 - w_2|^2 + \rho^2 C^2 |w_1 - w_2|^2 - 2\rho\alpha |w_1 - w_2|^2, \quad \forall w_1, w_2 \in K, \\ &\leq \left(1 + \rho^2 C^2 - 2\rho\alpha\right) |w_1 - w_2|^2, \quad \forall w_1, w_2 \in K \end{aligned}$$

Therefore, if we choose ρ such that

$$0 \leq 1 + \rho^2 C^2 - 2\rho\alpha < 1,$$

we conclude, by setting $\sqrt{1 + \rho^2 C^2 - 2\rho\alpha} = k$, that

$$|S(w_1) - S(w_2)| \leq k |w_1 - w_2|, \quad \forall w_1, w_2 \in K$$

which means that S is a contraction.

Thus, Banach fixed point theorem asserts that there exists a unique element $u \in K$ such that

$$S(u) = u.$$

By replacing w and $S(w)$ by u in (3.10) it becomes

$$\langle \rho f - \rho Au, v - u \rangle \leq 0, \quad \forall v \in K$$

and hence

$$\rho \langle f, v - u \rangle \leq \rho \langle Au, v - u \rangle , \quad \forall v \in K.$$

Therefore, since ρ is positive, we have

$$\langle f, v - u \rangle \leq \langle Au, v - u \rangle , \quad \forall v \in K.$$

This completes the proof of the theorem.

Remark : Besides the result aforementioned we can add the following one, where the proof can be found in [9].

If K is a closed convex cone with vertex 0, then

$$\begin{cases} a(u, v) \geq \varphi(v), \quad \forall v \in K \\ a(u, u) = \varphi(u), \end{cases}$$

Lemma 1 (Lax-Milgram lemma) :

Let $a(.,.)$ be a continuous bilinear and coercive form defined on H, then for every $\varphi \in H'$ there exists a unique $u \in H$ such that

$$a(u, v) = \varphi(v), \quad \forall v \in H.$$

Moreover, if $a(.,.)$ is symmetric, u is characterized by

$$\begin{cases} u \in H \\ \frac{1}{2} a(u, u) - \varphi(u) = \min_{v \in H} \left\{ \frac{1}{2} a(v, v) - \varphi(v) \right\} \end{cases}$$

Proof: By using Stampacchia theorem, there exists a unique $u \in H$ such that

$$a(u, v - u) \geq \varphi(v - u), \quad \forall v \in H,$$

since $tv \in H$ for every $t \in \mathbb{R}$, then replacing v by tv we get

$$a(u, tv-u) \geq \varphi(tv-u), \quad \forall v \in H, \quad \forall t \in \mathbb{R},$$

so,

$$t\{a(u,v)-\varphi(v)\} \geq \{a(u,u)-\varphi(u)\}, \quad \forall t \in \mathbb{R}, \quad \forall v \in H.$$

Suppose that $a(u,v)-\varphi(v) \neq 0$, then, we can make $a(u,u)-\varphi(u) \longrightarrow -\infty$ by making $t \longrightarrow +\infty$ or $t \longrightarrow -\infty$, thus, a contradiction.

Therefore,

$$a(u,v) = \varphi(v), \quad \forall x \in H.$$

Homogeneous Dirichlet problem

We are now ready to study the given problem (3.1).

Proposition 4 : Every classical solution to (3.1) is a weak solution.

Proof: Suppose that $u \in C^2\left(\overline{\Omega}\right)$ is a classical solution to (3.1).

Since Ω is bounded, then, from **proposition II-3**, we have $C^2\left(\overline{\Omega}\right) \subset H^1(\Omega)$, hence,

$$u \in H^1(\Omega) \cap C(\overline{\Omega}).$$

Furthermore, since $u|_{\partial\Omega} = 0$, we can use the trace theorem, to get

$$u \in H_0^1(\Omega).$$

On the other hand, multiplying both sides of the first equation in (3.1) by $v \in C_0^\infty(\Omega)$, integrating over Ω and using integration by parts, we arrive at

$$\sum_{i=1}^{N} \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} + \int_\Omega uv = \int_\Omega fv, \quad \forall v \in C_0^\infty(\Omega).$$

By density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$, this equality remains valid for every $v \in H_0^1(\Omega)$. Thus, u is a weak solution to (3.1).

Remark: In the above proof we used a result of the trace theorem in spite of the hypothesis that the boundary will be of class C^1 is not satisfied, because in the proof of the mentioned theorem, the fact that if $u|_{\partial\Omega} = 0$, then $u \in H_0^1(\Omega)$, doesn't use this hypothesis, see [5].

Theorem 5: For all $f \in L^2(\Omega)$, problem (3.1) has a unique weak solution.

To prove this theorem, we need to prove a lemma.

Lemma 2: For every $u, v \in H^1(\Omega)$ we have

$$\int_\Omega |\nabla u|\, |\nabla v| + |u|\, |v| \leq \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)},$$

where $\|u\|_{H^1(\Omega)}$ is the norm defined by

$$\|u\|_{H^1(\Omega)} = \|u\|_{L^2} + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2}.$$

Proof of lemma 2: Since

$$(|u|\, |\nabla v| - |v|\, |\nabla u|)^2 \geq 0, \ \ \forall u, v \in H^1(\Omega),$$

then,

$$|u|^2 |\nabla v|^2 + |v|^2 |\nabla u|^2 \geq 2\, |u|\, |\nabla v|\, |v|\, |\nabla u|,$$

consequently,

$$|u|^2 |v|^2 + |\nabla u|^2 |\nabla v|^2 + |u|^2 |\nabla v|^2 + |v|^2 |\nabla u|^2 \geq (|\nabla u|\, |\nabla v| + |u|\, |v|)^2,$$

thus,

$$\left(|u|^2 + |\nabla u|^2\right) \left(|v|^2 + |\nabla v|^2\right) \geq (|\nabla u|\, |\nabla v| + |u|\, |v|)^2,$$

then, by integration over Ω we have

$$\int_\Omega \left(|u|^2 + |\nabla u|^2\right)^{\frac{1}{2}} \left(|v|^2 + |\nabla v|^2\right)^{\frac{1}{2}} \geq \int_\Omega (|\nabla u|\, |\nabla v| + |u|\, |v|), \quad \forall u, v \in H^1(\Omega),$$

by using Cauchy Schwarz inequality, for the left-hand side, we have

$$\left(\int_\Omega \left(|u|^2 + |\nabla u|^2\right)\right)^{\frac{1}{2}} \left(\int_\Omega \left(|v|^2 + |\nabla v|^2\right)\right)^{\frac{1}{2}} \geq \int_\Omega (|\nabla u|\, |\nabla v| + |u|\, |v|), \ \forall u, v \in H^1(\Omega). \tag{3.11}$$

Also,

$$\begin{aligned}\|u\|_{H^1}^2 &= \left(\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^2} + \|u\|_{L^2}\right)^2 \\ &= \left(\sum_{i=1}^{N}\left(\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^2\right)^{\frac{1}{2}} + \left(\int_\Omega |u|^2\right)^{\frac{1}{2}}\right)^2;\end{aligned}$$

by developing of the square in last parenthesis we easily show that

$$\begin{aligned}\|u\|_{H^1}^2 &\geq \sum_{i=1}^{N}\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^2 + \int_\Omega |u|^2, \\ &\geq \int_\Omega |\nabla u|^2 + \int_\Omega |u|^2. \end{aligned} \tag{3.12}$$

Thus, from (3.11) and (3.12), we get

$$\|u\|_{H^1}\|v\|_{H^1} \geq \int_\Omega \left(|\nabla u|\,|\nabla v| + |u|\,|v|\right), \quad \forall u, v \in H^1(\Omega).$$

This completes the proof of **Lemma 2**.

Proof of theorem 5:

a) A bilinear form:

In the Hilbert space $H_0^1(\Omega)$, the form a defined by

$$a(u,v) = \sum_{i=1}^{N}\int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} + \int_\Omega uv,$$

is bilinear, continuous and coercive form.

Indeed,

1) **Continuity:** For every $u, v \in H_0^1(\Omega)$, we have

$$a(u,v) = \int_\Omega \nabla u.\nabla v + \int_\Omega uv,$$

then,

$$|a(u,v)| = \left|\int_\Omega \nabla u.\nabla v + \int_\Omega uv\right|$$

$$|a(u,v)| \leq \int_{\Omega} |\nabla u| \, |\nabla v| + |u| \, |v| \, , \quad \forall u, v \in H^1(\Omega),$$

by using **Lemma 2** we have

$$|a(u,v)| \leq \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \, , \quad \forall u, v \in H^1(\Omega),$$

therefore, since $H_0^1(\Omega) \subset H^1(\Omega)$ and H_0^1 norm is induced by H^1 norm, the last inequality holds in $H_0^1(\Omega)$; that is,

$$|a(u,v)| \leq \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} \, , \quad \forall u, v \in H_0^1(\Omega).$$

Thus, $a(.,.)$ is continuous.

2) **Coercivity**:

$$\begin{aligned} a(u,u) &= \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^2 + \int_{\Omega} u^2, \quad \forall u \in H_0^1(\Omega), \\ &= \int_{\Omega} |\nabla u|^2 + \int_{\Omega} |u|^2 \, , \quad \forall u \in H_0^1(\Omega), \\ &\geq \int_{\Omega} |\nabla u|^2 \, , \quad \forall u \in H_0^1(\Omega), \end{aligned}$$

then,

$$a(u,u) \geq \|\nabla u\|_{L^2}^2 \, , \quad \forall u \in H_0^1(\Omega).$$

Since Ω is bounded, $\|\nabla u\|_{L^2}$ define a norm in $H_0^1(\Omega)$ equivalent to the norm reduced by $H^1(\Omega)$ norm, thus, there exists a constant $\alpha > 0$ such that

$$a(u,u) \geq \alpha \|u\|_{H_0^1}^2 \, , \quad \forall u \in H_0^1(\Omega).$$

b) A linear form:

Let φ be the form defined on $H_0^1(\Omega)$ by

$$\varphi(v) = \int_{\Omega} fv, \quad \forall v \in H_0^1(\Omega),$$

then φ is a linear continuous form.

Indeed, it is easy to show that φ is linear.

Also, by using Cauchy Schwarz inequality

$$\begin{aligned}
|\varphi(v)| &\leq \left(\int_\Omega |f|^2\right)^{\frac{1}{2}} \left(\int_\Omega |v|^2\right)^{\frac{1}{2}}, \quad \forall v \in H_0^1(\Omega), \\
&\leq \|f\|_{L^2} \|v\|_{L^2}, \quad \forall v \in H_0^1(\Omega), \\
&\leq C \|v\|_{H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega).
\end{aligned}$$

where $C = \|f\|_{L^2}$. Thus, φ is continuous.

By using Lax-Milgram lemma for the bilinear form $a(.,.)$ and the linear form φ, problem (3.2) has a unique solution $u \in H_0^1(\Omega)$.

Moreover, the bilinear form $a(.,.)$ is symmetric, then u minimizes the functional

$$J(v) = \frac{1}{2} a(v,v) - \varphi(v),$$

in $H_0^1(\Omega)$, which is the Dirichlet principle.

3.3 Problem 2

Let L be the elliptic operator on the divergence form

$$L(u) = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + a_0(x)\,u,$$

where a_{ij} and a_0 are in $L^\infty(\Omega)$ and consider the problem:

Find u which satisfies,

$$\begin{cases} -\sum\limits_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + a_0(x)\,u = f, \quad \text{in } \Omega \\ u = 0, \text{ in } \partial\Omega \end{cases} \tag{3.13}$$

Definition: We say that the functions a_{ij} verify the coercivity property, if there exists a constant $\alpha > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}\xi_j\xi_i \geq \alpha \sum_{i=1}^{N} \xi_i^2, \quad \forall x \in \Omega \quad \text{and} \quad \forall \xi \in \mathbb{R}^N,$$

or, in the equivalent form

$$((a_{ij}) .\xi) .\xi \geq \alpha\xi.\xi, \quad \forall x \in \Omega, \quad \forall \xi \in \mathbb{R}^N.$$

Theorem 6: Suppose that the functions a_{ij} verifies the property of coercivity and $a_0(x) > 0$ in Ω, then, for every $f \in L^2(\Omega)$, problem (3.13) has a unique weak solution $u \in H_0^1(\Omega)$.

Proof: It's easy to show that every classical solution to (3.13) verifies

$$\sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \int_\Omega a_0 uv = \int_\Omega fv, \quad \forall v \in H_0^1(\Omega), \tag{3.14}$$

and then, it is a weak solution.

We define a bilinear form $a(.,.)$ on $H_0^1(\Omega)$ by

$$a(u,v) = \sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \int_\Omega a_0 uv,$$

and a linear form φ by

$$\varphi(v) = \int_\Omega fv,$$

then, $a(.,.)$ and φ verify the hypotheses of Lax-Milgram lemma.

Indeed,

1) $a(.,.)$ **is continuous**,

$$a(u,v) = \int_\Omega (a_{ij}(x) .\nabla u) .\nabla v + \int_\Omega a_0 uv,$$

we can show that,

$$|a(u,v)| \leq M \sum_{1 \leq i,j \leq N} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right| \left| \frac{\partial v}{\partial x_j} \right| + m \int_\Omega |u| \, |v|, \quad \forall u,v \in H_0^1(\Omega),$$

where $M = \sup\limits_{\substack{1 \leq i,j \leq N \\ x \in \Omega}} |a_{ij}(x)|$ and $m = \sup\limits_{x \in \Omega} |a_0(x)|$.

Consequently,

$$\begin{aligned} |a(u,v)| &\leq MN^2 \int_\Omega |\nabla u| |\nabla v| + m \int_\Omega |u| |v|, \quad \forall u, v \in H_0^1(\Omega), \\ &\leq \max\left\{MN^2, m\right\} \int_\Omega |\nabla u| |\nabla v| + |u| |v|, \quad \forall u, v \in H_0^1(\Omega), \end{aligned}$$

By using **Lemma** 2, we arrive at

$$|a(u,v)| \leq \max\left\{MN^2, m\right\} \|u\|_{H^1} \|v\|_{H^1}, \quad \forall u, v \in H_0^1(\Omega),$$

then, since $\|u\|_{H^1} = \|u\|_{H_0^1}$, for $u \in H_0^1(\Omega)$,

$$|a(u,v)| \leq \max\left\{MN^2, m\right\} \|u\|_{H_0^1} \|v\|_{H_0^1}, \quad \forall u, v \in H_0^1(\Omega),$$

Thus, there exists a positive constant $C = \max\left\{MN^2, m\right\}$ such that

$$|a(u,v)| \leq C \|u\|_{H_0^1} \|v\|_{H_0^1}, \quad \forall u, v \in H_0^1(\Omega).$$

Therefore, $a(.,.)$ is continuous.

2) $a(.,.)$ **is coercive**,

by using the coercivity property we have

$$\begin{aligned} a(u,u) &= \sum_{i,j=1}^N \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \int_\Omega a_0(x) u^2, \quad \forall u \in H_0^1(\Omega) \\ &\geq \alpha \sum_{i=1}^N \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^2 + \int_\Omega a_0 u^2, \quad \forall u \in H_0^1(\Omega), \end{aligned}$$

since a_0 is positive then,

$$a(u,u) \geq \alpha \|\nabla u\|_{L^2}^2, \quad \forall u \in H_0^1(\Omega),$$

from **corollary II-14**, $\|\nabla u\|_{L^2}$ define an equivalent norm in $H_0^1(\Omega)$. Then,

$$a(u,u) \geq \alpha \|u\|_{H_0^1}^2, \quad \forall u \in H_0^1(\Omega),$$

which completes the proof of the coercivity of $a(.,.)$.

Remark : In what is above we denote $\|\nabla u\|_{(L^2)^N}$ by $\|\nabla u\|_{L^2}$.

3) φ is continuous, as in N° 3 of the proof of theorem 5.

Thus, by using Lax-Milgram lemma there exists a unique solution $u \in H_0^1(\Omega)$ to problem (3.14), which is a weak solution to (3.13).

3.4 Problem 3: Nonhomogeneous Neumann problem

Consider the problem

$$\begin{cases} -\Delta u + a_0 u = f \text{ in } \Omega \\ \frac{\partial u}{\partial \eta} = g \text{ in } \partial\Omega \end{cases} \tag{3.15}$$

where $a_0 \in L^\infty(\Omega)$, $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$.

Multiplying both sides by $v \in C^1(\Omega) \cap H^1(\Omega)$ and integrating over Ω, by using Green's formula, we arrive at

$$\sum_{i=1}^{N} \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} dx - \int_{\partial\Omega} \frac{\partial u}{\partial \eta} v d\sigma + \int_\Omega a_0 u v dx = \int_\Omega f v dx,$$

then,

$$\sum_{i=1}^{N} \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} dx + \int_\Omega a_0 u v dx = \int_\Omega f v dx + \int_{\partial\Omega} \frac{\partial u}{\partial \eta} v d\sigma, \ \ \forall v \in C^1(\Omega) \cap H^1(\Omega), \tag{3.16}$$

since $C^1(\Omega) \cap H^1(\Omega)$ is dense in $H^1(\Omega)$, the last equality holds for every $v \in H^1(\Omega)$.

Definition : We say that a function $u \in H^1(\Omega)$ is a weak solution to the problem (3.15) if u verifies (3.16) $\forall v \in H^1(\Omega)$.

Study of Problem 3:

Let $a(.,.)$ be the bilinear form defined on $H^1(\Omega)$ by

$$a\left(u,v\right)=\sum_{i=1}^{N}\int_{\Omega}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx+\int_{\Omega}a_0uvdx,\quad \forall u,v\in H^1\left(\Omega\right)$$

and φ be the linear form defined by

$$\varphi\left(v\right)=\int_{\Omega}fvdx+\int_{\partial\Omega}\frac{\partial u}{\partial\eta}vd\sigma,\quad \forall v\in H^1(\Omega).$$

Theorem 6: In addition to the assumptions on a_0, f and g, suppose that there exists a constant $\alpha_0>0$ such that

$$a_0\left(x\right)\geq\alpha_0,\ \text{almost everywhere in }\Omega.$$

Then problem (3.15) has a unique weak solution in $H^1\left(\Omega\right)$.

Proof : It suffices to prove that a and φ verify the hypotheses of Lax-Milgram lemma.

1) **Continuity of** $a\left(.,.\right)$

$$\left|a\left(u,v\right)\right|\leq\int_{\Omega}\left|\nabla u\right|\left|\nabla v\right|+\int_{\Omega}\left|a_0uv\right|,\quad \forall u,v\in H^1(\Omega),$$

$$\left|a\left(u,v\right)\right|\leq\int_{\Omega}\left|\nabla u\right|\left|\nabla v\right|+\left\|a_0\right\|_{\infty}\int_{\Omega}\left|uv\right|,\quad \forall u,v\in H^1(\Omega),$$

therefore,

$$\left|a\left(u,v\right)\right|\leq\max\left\{1,\left\|a_0\right\|_{\infty}\right\}\left(\int_{\Omega}\left|\nabla u\right|\left|\nabla v\right|+\int_{\Omega}\left|uv\right|\right),\quad \forall u,v\in H^1(\Omega),$$

by using **Lemma 2** we get

$$\left|a\left(u,v\right)\right|\leq\max\left\{1,\left\|a_0\right\|_{\infty}\right\}\left\|u\right\|_{H^1(\Omega)}\left\|v\right\|_{H^1(\Omega)},\quad \forall u,v\in H^1(\Omega),$$

then, for $C=\max\left\{1,\left\|a_0\right\|_{\infty}\right\}$ the last inequality takes the form

$$\left|a\left(u,v\right)\right|\leq C\left\|u\right\|_{H^1}\left\|v\right\|_{H^1},\quad \forall u,v\in H^1(\Omega).$$

Thus, $a\left(.,.\right)$ is continuous.

2) **Coercivity of** $a(.,.)$

$$\begin{aligned} a(u,u) &= \sum_{i=1}^{N}\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^2+\int_\Omega a_0(x)\,|u|^2, \quad \forall u\in H^1(\Omega),\\ &\geq \int_\Omega|\nabla u|^2+\alpha_0\int_\Omega u^2, \quad \forall u\in H^1(\Omega),\\ &\geq \min\{1,\alpha_0\}\left(\int_\Omega|\nabla u|^2+\int_\Omega|u|^2\right), \quad \forall u\in H^1(\Omega), \end{aligned}$$

therefore, using the equivalent between the norms

$$\|u\|_{H^1(\Omega)}=\left(\int_\Omega|\nabla u|^2+\int_\Omega u^2\right)^{\frac{1}{2}}$$

and

$$\|u\|_{H^1(\Omega)}=\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^2}+\|u\|_{L^2}$$

in $H^1(\Omega)$, we assert that there exists a constant $\beta>0$ such that

$$\begin{aligned} a(u,u) &\geq \min\{1,\alpha_0\}\left(\|\nabla u\|_{L^2}^2+\|u\|_{L^2}^2\right), \quad \forall u\in H^1(\Omega),\\ &\geq \beta\left(\sum_{i=1}^{n}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^2}+\|u\|_{L^2}\right)^2, \quad \forall u\in H^1(\Omega), \end{aligned}$$

then,

$$a(u,u)\geq\beta\,\|u\|_{H^1(\Omega)}^2, \quad \forall u\in H^1(\Omega).$$

Thus, $a(.,.)$ is coercive.

3) **Continuity of** φ: Recall that from the trace theorem, there exists a constant $B>0$ such that

$$\|v\|_{L^2(\partial\Omega)}\leq B\,\|v\|_{H^1(\Omega)}, \quad \forall v\in H^1(\Omega).$$

Therefore,

$$\begin{aligned}
|\varphi(v)| &\leq \int_\Omega |fv| + \int_{\partial\Omega} |gv|, \quad \forall v \in H^1(\Omega), \\
&\leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \|v\|_{L^2(\partial\Omega)}, \quad \forall v \in H^1(\Omega), \\
&\leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + B\|g\|_{L^2(\partial\Omega)} \|v\|_{H^1(\Omega)}, \quad \forall v \in H^1(\Omega),
\end{aligned}$$

hence,

$$|\varphi(v)| \leq \left(\|f\|_{L^2(\Omega)} + B\|g\|_{L^2(\partial\Omega)}\right) \|v\|_{H^1(\Omega)}, \quad \forall v \in H^1(\Omega).$$

Set $B' = \|f\|_{L^2(\Omega)} + B\|g\|_{L^2(\partial\Omega)}$, then,

$$|\varphi(v)| \leq B' \|v\|_{H^1(\Omega)}, \quad \forall v \in H^1(\Omega).$$

Thus, φ is continuous.

By Lax-Milgram lemma, problem (3.15) has a unique solution $u \in H^1(\Omega)$.

3.5 Problem 4: A nonsymmetric case

Let us consider the problem

$$\begin{cases} -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{N} b_i(x)\frac{\partial u}{\partial x_i} + a_0(x)u = f \text{ in } \Omega \\ u = 0 \text{ in } \partial\Omega \end{cases} \tag{3.17}$$

where $a_{ij}, b_i, a_0 \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$.

As in Dirichlet problem, multiplying the equation by $v \in C_0^\infty(\Omega)$ and integrating over Ω, using Green's formula, we have

$$\sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + \sum_{i=1}^{N} \int_\Omega b_i(x)\frac{\partial u}{\partial x_i} v + \int_\Omega a_0(x) uv = \int_\Omega fv, \ \forall v \in C_0^\infty(\Omega).$$

By density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$, the above equality holds for every $v \in H_0^1(\Omega)$ and a function $u \in H_0^1(\Omega)$. So we get

$$\sum_{i,j=1}^{N}\int_{\Omega}a_{ij}\left(x\right)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}+\sum_{i=1}^{N}\int_{\Omega}b_i\left(x\right)\frac{\partial u}{\partial x_i}v+\int_{\Omega}a_0\left(x\right)uv=\int_{\Omega}fv,\ \ \forall v\in H_0^1(\Omega). \qquad (3.18)$$

In this case, u is said to be a weak solution of problem 4.

Let a_{ij}, b_i, a_0 belong to $L^{\infty}\left(\Omega\right)$ and $f\in L^2\left(\Omega\right)$.

Theorem 7: Suppose that a_{ij} verify the coercivity property. Then, there exists a constant $\gamma>0$ such that, if $a_0\left(x\right)\geq\gamma$ a.e. in Ω, the problem (3.17) has a unique weak solution $u\in H_0^1\left(\Omega\right)$ [14].

Proof : Define a bilinear form $a\left(.,.\right)$ and a linear form φ on $H_0^1\left(\Omega\right)$ by

$$a\left(u,v\right)=\sum_{i,j=1}^{N}\int_{\Omega}a_{ij}\left(x\right)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}+\sum_{i=1}^{N}\int_{\Omega}b_i\left(x\right)\frac{\partial u}{\partial x_i}v+\int_{\Omega}a_0\left(x\right)uv,$$

and

$$\varphi\left(v\right)=\int_{\Omega}fv$$

It suffices to prove that $a\left(.,.\right)$ and φ verify the hypothesis of Lax-Milgram lemma.

1) **Continuity of** $a\left(.,.\right)$:

$$\left|a\left(u,v\right)\right|\leq M\sum_{i,j=1}^{N}\int_{\Omega}\left|\frac{\partial u}{\partial x_j}\right|\left|\frac{\partial v}{\partial x_i}\right|+M_0\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial u}{\partial x_i}\right|\left|v\right|+m\int_{\Omega}\left|u\right|\left|v\right| \qquad (3.19)$$

where $M=\max\limits_{1\leq i,j\leq n}\left\|a_{ij}\right\|_{\infty}$, $\quad M_0=\max\limits_{1\leq i\leq n}\left\|b_i\right\|_{\infty}$ and $m=\left\|a_0\right\|_{\infty}$.

By using Cauchy Schwarz inequality, (3.19) can be written

$$\begin{aligned}\left|a\left(u,v\right)\right| \leq\ & M\sum_{i,j=1}^{N}\left(\int_{\Omega}\left|\frac{\partial u}{\partial x_j}\right|^2\right)^{\frac{1}{2}}\left(\int_{\Omega}\left|\frac{\partial v}{\partial x_i}\right|^2\right)^{\frac{1}{2}}+M\sum_{i=1}^{N}\left(\int_{\Omega}\left|\frac{\partial u}{\partial x_i}\right|^2\right)^{\frac{1}{2}}\left(\int_{\Omega}\left|v\right|^2\right)^{\frac{1}{2}}\\ & +m\left(\int_{\Omega}\left|u\right|^2\right)^{\frac{1}{2}}\left(\int_{\Omega}\left|v\right|^2\right)^{\frac{1}{2}},\\ & M\left(\sum_{j=1}^{N}\left\|\frac{\partial u}{\partial x_j}\right\|_{L^2}\right)\left(\sum_{i=1}^{N}\left\|\frac{\partial v}{\partial x_i}\right\|_{L^2}\right)+M_0\left(\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^2}\right)\left\|v\right\|_{L^20}\\ & +m\left\|u\right\|_{L^2}\left\|v\right\|_{L^2}\end{aligned}$$

$$
\begin{aligned}
|a(u,v)| &\leq M_1 \left(\sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2} + \|u\|_{L^2} \right) \left(\sum_{j=1}^{N} \left\| \frac{\partial v}{\partial x_j} \right\|_{L^2} + \|v\|_{L^2} \right), \quad \forall u, v \in H_0^1(\Omega). \\
&\leq M_1 \|u\|_{H^1} \|v\|_{H^1}, \quad \forall u, v \in H_0^1(\Omega),
\end{aligned}
$$

where $M_1 = \max\{M, M_0, m\}$.

Thus, $a(.,.)$ is continuous.

2) **Coercivity**:

$$
a(u,u) = \sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} + \sum_{i=1}^{N} \int_\Omega b_i(x) \frac{\partial u}{\partial x_i} u + \int_\Omega a_0(x) u^2, \quad \forall u \in H_0^1(\Omega),
$$

by the coercivity property we have

$$
a(u,u) \geq \alpha \int_\Omega |\nabla u|^2 + \sum_{i=1}^{N} \int_\Omega b_i(x) \frac{\partial u}{\partial x_i} u + \int_\Omega a_0(x) u^2, \quad \forall u \in H_0^1(\Omega)
$$

it is easy to show that

$$
\begin{aligned}
a(u,u) - \int_\Omega a_0(x) u^2 &\geq \alpha \int_\Omega |\nabla u|^2 - \sum_{i=1}^{N} \int_\Omega \left| b_i(x) \frac{\partial u}{\partial x_i} u \right| \\
a(u,u) - \int_\Omega a_0(x) u^2 &\geq \alpha \int_\Omega |\nabla u|^2 - \max_{1 \leq i \leq N} \|b_i\|_\infty \sum_{i=1}^{N} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right| |u|, \qquad (3.20)
\end{aligned}
$$

recall that from Young's inequality we have

$$
\int_\Omega \left| \frac{\partial u}{\partial x_i} \right| |u| \leq \varepsilon \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^2 + \frac{1}{4\varepsilon} \int_\Omega |u|^2, \quad \forall \varepsilon > 0, \quad \forall u \in H_0^1(\Omega),
$$

if we insert this estimate of $\int_\Omega \left| \frac{\partial u}{\partial x_i} \right| |u|$ in (3.20) we have

$$
\begin{aligned}
a(u,u) - \int_\Omega a_0(x) u^2 &\geq \alpha \int_\Omega |\nabla u|^2 - M_0 \varepsilon \int_\Omega |\nabla u|^2 - \frac{M_0 N}{4\varepsilon} \int_\Omega |u|^2, \ \forall \varepsilon > 0, \ \forall u \in H_0^1(\Omega), \\
&\geq (\alpha - M_0 \varepsilon) \int_\Omega |\nabla u|^2 - \frac{M_0 N}{4\varepsilon} \int_\Omega |u|^2, \ \forall \varepsilon > 0, \ \forall u \in H_0^1(\Omega),
\end{aligned}
$$

then for $\varepsilon < \frac{\alpha}{M_0}$ we have $\alpha - M_0\varepsilon > 0$ and

$$a(u,u) \geq (\alpha - M_0\varepsilon) \int_\Omega |\nabla u|^2 + \int_\Omega \left(a_0(x) - \frac{M_0 N}{4\varepsilon} \right) |u|^2, \quad \forall u \in H_0^1(\Omega). \tag{3.21}$$

To get $a_0(x) - \frac{M_0 N}{4\varepsilon} \geq 0$ it suffices that a_0 satisfy

$$a_0(x) \geq \frac{M_0 N}{4\varepsilon},$$

so, from the above estimate for ε, the estimate for a_0 becomes

$$a_0(x) > \frac{M_0^2 N}{4\alpha} \text{ for a.e. } x \in \Omega.$$

Then, if $a_0(x) \geq \gamma$ a.e. in Ω, for any $\gamma > \frac{M_0^2 N}{4\alpha}$, it suffices to choose ε such that

$$\gamma \geq \frac{M_0 N}{4\varepsilon} > \frac{M_0^2 N}{4\alpha}$$

to get

$$a_0(x) - \frac{M_0 N}{4\varepsilon} \geq 0$$

and (3.21) implies that

$$a(u,u) \geq (\alpha - M_0\varepsilon) \int_\Omega |\nabla u|^2, \quad \forall u \in H_0^1(\Omega).$$

Thus, because $\int_\Omega |\nabla u|^2$ define a norm in $H_0^1(\Omega)$, the last inequality can be written

$$a(u,u) \geq \alpha' \|u\|^2_{H_0^1(\Omega)},$$

for some positive constant α'. Therefore, $a(.,.)$ is coercive in $H_0^1(\Omega)$.

3) **Continuity of** φ: As in problem 1.

We have showed that $a(.,.)$ and φ fulfilled the hypotheses of Lax-Milgram lemma. Thus, problem (3.18) has a unique solution $u \in H_0^1(\Omega)$.

Chapter 4

Nonlinear problems

4.1 First Problem

Let Ω be an open bounded subset of $\mathbb{R}^N$ with a boundary $\partial\Omega$. We consider the following nonlinear elliptic boundary value problem

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a\left(x,u\right)\frac{\partial u}{\partial x_i}\right) = f \ \text{ in } \Omega \\ u = 0 \ \text{ in } \partial\Omega \end{cases} \tag{4.1}$$

where f is a given function to be specified later.

To get the weak formulation of problem (4.1) we multiply both sides of the equation (4.1) by $v \in C_0^\infty\left(\Omega\right)$, integrate over Ω and use the integration by parts to get

$$\begin{cases} \sum_{i=1}^{N} \int_\Omega a\left(x,u\right)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} = \int_\Omega fv, \quad \forall v \in C_0^\infty\left(\Omega\right) \\ u = 0 \ \text{ in } \partial\Omega \end{cases} \tag{4.2}$$

by using the density of $C_0^\infty\left(\Omega\right)$ in $H_0^1\left(\Omega\right)$, (4.2) holds for every $v \in H_0^1\left(\Omega\right)$. Then, a weak solution to (4.1) is a function $u \in H_0^1\left(\Omega\right)$ which satisfies

$$\sum_{i=1}^{N}\int_\Omega a\left(x,u\right)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} = \int_\Omega fv, \quad \forall v \in H_0^1(\Omega). \tag{4.3}$$

Definition : Let a be the function $a : \Omega \times \mathbb{R} \to \mathbb{R}$.

We say that a is a Caratheodory function, if

$i)$ for every $t \in \mathbb{R}$, the function $a(.,t) : \Omega \to \mathbb{R}$ is measurable,

$ii)$ for almost everywhere $x \in \Omega$, the function $a(x,.) : \mathbb{R} \to \mathbb{R}$ is continuous.

Theorem 1: Suppose that a is a Caratheodory function and that there exist two constants m and M such that

$$0 < m \leq a(x,t) \leq M, \text{ for a.e. } x \in \Omega \text{ and } \forall t \in \mathbb{R}.$$

Then, for every $f \in H^{-1}(\Omega)$, the problem (4.2) has a solution $u \in H_0^1(\Omega)$.

Proof :

To study this problem we use a technique frequently used for nonlinear partial differential equations. We will use a priori estimate for the solution of such problem and to do so one can use a fixed point theorem to solve approximative problems in finite dimensional spaces where one can obtain various results, then, one has to pass to the limit in the dimensional by using a compact embedding theorem of Sobolev.

$H_0^1(\Omega)$ is a separable Hilbert space, so it has a Hilbertian basis [5], that is to say, there exists a sequence

$$\{e_n; n \in \mathbb{N}^*\} \subset H_0^1(\Omega),$$

such that

$$\langle e_n, e_m \rangle = \delta_{nm}, \quad \forall n, m \in \mathbb{N}^*,$$

and the space generated by $\{e_n; n \in \mathbb{N}^*\}$ is dense in $H_0^1(\Omega)$, where δ_{nm} is the Kronecker delta.

Also,

$$u = \sum_{k=1}^{\infty} \langle u, e_k \rangle e_k, \quad \forall u \in H_0^1(\Omega)$$

and

$$\|u\|_{H_0^1(\Omega\ )}^2 = \sum_{k=1}^{\infty} \langle u, e_k \rangle^2, \quad \forall u \in H_0^1(\Omega).$$

Let $V_n = Span[e_1, e_2, ..., e_n]$; that is the space generated by $\{e_1, e_2, ..., e_n\}$.

In each subspace V_n we consider an approximate problem to the given problem (4.2),

$$\begin{cases} u_n \in V_n \\ \sum_{i=1}^{N} \int_\Omega a(x, u_n) \frac{\partial u_n}{\partial x_i} \frac{\partial v}{\partial x_i} = \int_\Omega fv, \quad \forall v \in V_n \end{cases} \tag{4.4}$$

it is a nonlinear problem in a finite dimensional space V_n.

A linear approximate problem:

Let $w \in V_n$ and change the problem (4.4) to a linear one

$$\begin{cases} u_w \in V_n \\ \sum_{i=1}^{N} \int_\Omega a(x, w) \frac{\partial u_w}{\partial x_i} \frac{\partial v}{\partial x_i} = \int_\Omega fv, \quad \forall v \in V_n. \end{cases} \tag{4.5}$$

Proposition 2: For every $f \in H^{-1}(\Omega)$, problem (4.5) has a unique solution $u_w \in V_n$.

Proof: Note first that, since V_n is a finite-dimensional space, all the norms are equivalent, then, we equip V_n by the norm induced by $H_0^1(\Omega)$ norm, i.e.

$$\|u\|_{V_n} = \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2(\Omega)}.$$

Let $b(.,.)$ be the bilinear form defined in V_n by

$$b(u, v) = \sum_{i=1}^{N} \int_\Omega a(x, w) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}, \quad \forall u, v \in V_n$$

and let φ be the linear form defined by

$$\varphi(v) = \int_\Omega fv, \quad \forall v \in V_n.$$

To use the Lax-Milgram lemma we must prove that $b(.,.)$ and φ verify the hypotheses of **lemma 1 in chapter 2**.

1) b **is continuous,**

$$
\begin{aligned}
|b(u,v)| &\leq \sum_{i=1}^{N} \int_{\Omega} |a(x,w)| \left| \frac{\partial u}{\partial x_i} \right| \left| \frac{\partial v}{\partial x_i} \right|, \quad \forall u, v \in V_n \\
&\leq M \sum_{i=1}^{N} \int_{\Omega} |\nabla u| \, |\nabla v| \,, \quad \forall u, v \in V_n, \\
&\leq MN \int_{\Omega} |\nabla u| \, |\nabla v| \,, \quad \forall u, v \in V_n,
\end{aligned}
$$

by using Cauchy Schwarz inequality for the right-hand side we get

$$
\begin{aligned}
|b(u,v)| &\leq MN \left(\int_{\Omega} |\nabla u|^2 \right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla v|^2 \right)^{\frac{1}{2}}, \quad \forall u, v \in V_n, \\
&\leq MN \, \|u\|_{H_0^1} \, \|v\|_{H_0^1} \,, \quad \forall u, v \in V_n,
\end{aligned}
$$

hence,

$$
|b(u,v)| \leq C \, \|u\|_{V_n} \, \|v\|_{V_n} \,, \quad \forall u, v \in V_n,
$$

where $C = MN$. Thus, b is continuous in V_n.

2) b **is coercive**,

$$
\begin{aligned}
b(u,u) &= \sum_{i=1}^{N} \int_{\Omega} a(x,w) \left| \frac{\partial u}{\partial x_i} \right|^2, \quad \forall u \in V_n, \\
&\geq m \int_{\Omega} |\nabla u|^2 \,, \quad \forall u \in V_n.
\end{aligned}
$$

Since Ω is bounded, $\int_{\Omega} |\nabla u|^2$ defines an equivalent norm to $\|u\|_{H_0^1}$ in V_n, then, the last inequality takes the form

$$
b(u,u) \geq m \, \|u\|_{V_n}^2 \,, \quad \forall u \in V_n.
$$

Therefore, b is coercive.

3) φ **is continuous,**

$$
\begin{aligned}
|\varphi(v)| &= |\langle f, v \rangle| \\
&\leq \|f\|_{H^{-1}} \, \|v\|_{H_0^1} \,, \quad \forall v \in H_0^1(\Omega),
\end{aligned}
$$

then, by setting $C' = \|f\|_{H^{-1}}$ we have

$$|\varphi(v)| \leq C' \|v\|_{H_0^1}, \quad \forall v \in H_0^1(\Omega),$$

which means that φ is continuous in $H_0^1(\Omega)$, consequently in V_n.

Thus, by using the Lax-Milgram lemma in V_n, we assert that there exists a unique solution u_w to problem (4.5)

$$b(u_w, v) = \varphi(v), \quad \forall v \in V_n.$$

Furthermore, if we replace v by u_w in the last equality, we have

$$b(u_w, u_w) = \varphi(u_w),$$

then, by using the coercivity property of b and the continuity of φ, we get

$$\begin{aligned} m \|u_w\|_{V_n}^2 &\leq b(u_w, u_w) \\ &\leq \varphi(u_w) \\ &\leq \|f\|_{H^{-1}} \|u_w\|_{V_n}, \end{aligned} \tag{4.6}$$

so, if $\|u_w\|_{V_n} \neq 0$, by dividing (4.6) by $\|u_w\|_{V_n}$, we obtain the estimation

$$\|u_w\|_{V_n} \leq \frac{\|f\|_{H^{-1}}}{m}, \tag{4.7}$$

which holds even if $\|u_w\|_{V_n} = 0$.

Remark: Note that V_n is a Hilbert space equipped by the inner product induced by H^1 inner product, so one can apply the Lax-Milgram lemma.

Let T be the mapping defined in V_n by

$$T : w \to u_w,$$

where w and u_w are those mentioned above.

Provided we choose w in the ball $B\left(0, \frac{\|f\|_{H^{-1}}}{m}\right) \subset V_n$, the solution $u_w = T(w)$ will be also in this ball. Therefore, by the Brouwer fixed point theorem, the mapping T has a fixed point in V_n, provided we can prove its continuity.

Lemma 1 : T is continuous.

Proof : Let $\{w_p\}$ be a convergent sequence in V_n such that

$$w_p \leq \frac{\|f\|_{H^{-1}}}{m}, \quad \forall p \in \mathbb{N}$$

and let $w \in V_n$ be its limit in V_n, i.e.

$$w_p \to w \text{ with } H_0^1\text{-norm.}$$

To prove the continuity of T it suffices to prove that

$$T(w_p) \to T(w) \quad \text{in } V_n.$$

Let u_p be the solution to (4.5) associated to w_p. From (4.7) we have

$$u_p \leq \frac{\|f\|_{H^{-1}}}{m}, \quad \forall p \in \mathbb{N},$$

then, the sequence $\{u_p\}$ is bounded in V_n which is of finite dimension. Thus, we can extract a convergent subsequence $\{u_{p_k}\}$. That is,

$$u_{p_k} \to u, \quad \text{in } V_n,$$

see [22].

Let $\{w_{p_k}\}$ be the subsequence extracted from $\{w_p\}$, it is a convergent sequence to w in $H_0^1(\Omega)$, consequently in $L^2(\Omega)$,

$$w_{p_k} \to w \text{ in } L^2(\Omega).$$

Therefore, from $\{w_{p_k}\}$, we can again extract a subsequence, which we still denoted $\{w_{p_k}\}$, such that

$$w_{p_k} \to w \text{ almost everywhere in } \Omega,$$

see [5]. Thus, there exists $\Omega' \subset \Omega$ such that

$$w_{p_k}(x) \to w(x), \quad \forall x \in \Omega'$$

where

$$meas\left(\Omega/\Omega'\right) = 0.$$

Recall that for almost everywhere $x \in \Omega'$, $a(x,.)$ is continuous, then, there exists $\Omega'' \subset \Omega'$ such that

$$meas\left(\Omega'/\Omega''\right) = 0$$

and

$$a(x,.) : t \to a(x,t) \text{ is continuous } \forall x \in \Omega''.$$

Thus,

$$a\left(x, w_{p_k}(x)\right) \to a\left(x, w(x)\right), \quad \forall x \in \Omega'',$$

also, since

$$meas\left(\Omega/\Omega''\right) \leq meas\left(\Omega/\Omega'\right) + meas\left(\Omega'/\Omega''\right)$$

then, $meas\left(\Omega/\Omega''\right) = 0$ and

$$a\left(x, w_{p_k}(x)\right) \to a\left(x, w(x)\right) \text{ almost everywhere in } \Omega. \tag{4.8}$$

For any $v \in V_n$, multiply both sides of (4.8) by $\frac{\partial v(x)}{\partial x_i}$ to get

$$a\left(x, w_{p_k}(x)\right) \frac{\partial v(x)}{\partial x_i} \to a\left(x, w(x)\right) \frac{\partial v(x)}{\partial x_i} \text{ a.e. in } \Omega, \tag{4.9}$$

furthermore,

$$\left| a\left(x, w_{p_k}(x)\right) \frac{\partial v(x)}{\partial x_i} \right| \leq M \left| \frac{\partial v(x)}{\partial x_i} \right| \text{ a.e. in } \Omega.$$

In the other hand, $v \in H_0^1(\Omega)$, gives,

$$M \left| \frac{\partial v(x)}{\partial x_i} \right| \in L^2(\Omega).$$

Then, the sequence $\left\{a\left(x, w_{n_k}\right) \frac{\partial v}{\partial x_i}\right\}$, $a\left(x, w\right) \frac{\partial v}{\partial x_i}$ and $M\left|\frac{\partial v}{\partial x_i}\right|$ satisfy the hypotheses of the dominated convergence theorem.

Thus,

$$a\left(x, w_{p_k}\right) \frac{\partial v}{\partial x_i}, \quad a\left(x, w\right) \frac{\partial v}{\partial x_i} \in L^2\left(\Omega\right)$$

and

$$a\left(x, w_{p_k}\right) \frac{\partial v}{\partial x_i} \longrightarrow a\left(x, w\right) \frac{\partial v}{\partial x_i} \text{ in } L^2\left(\Omega\right). \tag{4.10}$$

In the other hand we have

$$u_{p_k} \longrightarrow u \text{ in } V_n,$$

which implies that

$$\frac{\partial u_{p_k}}{\partial x_i} \longrightarrow \frac{\partial u}{\partial x_i} \text{ in } L^2\left(\Omega\right).$$

Set $f_{p_k} = a\left(x, w_{p_k}\right) \frac{\partial v}{\partial x_i}, \quad f = a\left(x, w\right) \frac{\partial v}{\partial x_i}, \quad g_{p_k} = \frac{\partial u_{p_k}}{\partial x_i}$ and $g = \frac{\partial u}{\partial x_i}$, by using the fact that if

$$f_{p_k} \rightarrow f \text{ in } L^2\left(\Omega\right)$$

and

$$g_{p_k} \rightarrow g \text{ in } L^2\left(\Omega\right)$$

then

$$f_{p_k} g_{p_k} \rightarrow fg \text{ in } L^1(\Omega),$$

we arrive at

$$a\left(x, w_{p_k}\right) \frac{\partial u_{p_k}}{\partial x_i} \frac{\partial v}{\partial x_i} \longrightarrow a\left(x, w\right) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \text{ in } L^1(\Omega),$$

after a summation over i, we get

$$\sum_{i=1}^{N} \int_{\Omega} a\left(x, w_{p_k}\right) \frac{\partial u_{p_k}}{\partial x_i} \frac{\partial v}{\partial x_i} \rightarrow \sum_{i=1}^{N} \int_{\Omega} a\left(x, w\right) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}, \quad \forall v \in V_n. \tag{4.11}$$

Recall that u_{p_k} is the solution to (4.5) associated to w_{p_k}, so, for the left hand-side we have

$$\sum_{i=1}^{N} \int_{\Omega} a\left(x, w_{p_k}\right) \frac{\partial u_{p_k}}{\partial x_i} \frac{\partial v}{\partial x_i} = \int_{\Omega} fv, \quad \forall v \in V_n.$$

Passing to the limit in the last equality and using (4.11) we arrive at

$$\sum_{i=1}^{N} \int_{\Omega} a\left(x, w\right) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} = \int_{\Omega} fv, \quad \forall v \in V_n, \tag{4.12}$$

which implies that u is the solution to (4.5) corresponding to w, hence,

$$u = T(w),$$

which proves that T is continuous. Then, by using the Brouwer fixed point theorem, we conclude that there exists $u_n \in B\left(0, \frac{\|f\|_{H^{-1}}}{m'}\right) \subset V_n$ such that

$$u_n = T(u_n).$$

Thus, the corresponding problem (4.12) to $w = u_n$, has u_n as a solution in V_n, this fact can be written by

$$\sum_{i=1}^{N} \int_{\Omega} a\left(x, u_n\right) \frac{\partial u_n}{\partial x_i} \frac{\partial v}{\partial x_i} = \int_{\Omega} fv, \quad \forall v \in V_n. \tag{4.13}$$

Therefore, u_n is a solution to (4.4) in V_n.

4.2 The nonlinear problem in $H_0^1(\Omega)$

Now one would like to show that at the limit u_n will provide us with a solution to the given problem (4.3).

Let $\{u_n\}$ be the sequence in $H_0^1(\Omega)$ constructed by choosing in each V_n the solution to the nonlinear problem (4.13), then, $\{u_n\}$ is bounded in $H_0^1(\Omega)$,

$$\|u_n\|_{H_0^1} \leq \frac{\|f\|_{H^{-1}}}{m}.$$

By using the compact embedding of $H_0^1(\Omega)$ in $L^2(\Omega)$ we can extract a subsequence of $\{u_n\}$, which we still denote $\{u_n\}$, such that there exists $u \in H_0^1(\Omega)$ and

$$\begin{aligned} u_n &\rightharpoonup u \text{ in } H_0^1(\Omega), \\ u_n &\rightarrow u \text{ in } L^2(\Omega) \end{aligned} \tag{4.14}$$

and

$$u_n \rightarrow u \text{ a.e. in } \Omega,$$

see [3], [20] and [5] respectively.

Thus, by using (4.8) we have

$$a(x, u_n(x)) \rightarrow a(x, u(x)) \quad \text{a.e. } x \in \Omega.$$

In the other hand, let v be in $H_0^1(\Omega)$ and $\{v_n\} \subset H_0^1(\Omega)$ be a convergent sequence to v in $H_0^1(\Omega)$, then,

$$\frac{\partial v_n}{\partial x_i} \rightarrow \frac{\partial v}{\partial x_i} \quad \text{in } L^2(\Omega). \tag{4.15}$$

We want to show that

$$a(x, u_n(x)) \frac{\partial v_n}{\partial x_i} \rightarrow a(x, u(x)) \frac{\partial v}{\partial x_i} \quad \text{in } L^2(\Omega),$$

To do this, we use Minkowski's inequality to get

$$\left(\int_\Omega \left| a(x, u_n) \frac{\partial v_n}{\partial x_i} - a(x, u) \frac{\partial v}{\partial x_i} \right|^2 \right)^{\frac{1}{2}} \leq$$

$$\left(\int_\Omega \left| a(x, u_n) \frac{\partial v_n}{\partial x_i} - a(x, u_n) \frac{\partial v}{\partial x_i} \right|^2 \right)^{\frac{1}{2}} + \left(\int_\Omega \left| a(x, u_n) \frac{\partial v}{\partial x_i} - a(x, u) \frac{\partial v}{\partial x_i} \right|^2 \right)^{\frac{1}{2}}. \tag{4.16}$$

Note that

$$\left(\int_\Omega \left| a(x, u_n) \frac{\partial v_n}{\partial x_i} - a(x, u_n) \frac{\partial v}{\partial x_i} \right|^2 \right)^{\frac{1}{2}} \leq M \left(\int_\Omega \left| \frac{\partial v_n}{\partial x_i} - \frac{\partial v}{\partial x_i} \right|^2 \right)^{\frac{1}{2}}.$$

So, by (4.15) we conclude that

$$\left(\int_\Omega \left| a\left(x,u_n\right)\frac{\partial v_n}{\partial x_i} - a\left(x,u_n\right)\frac{\partial v}{\partial x_i}\right|^2\right)^{\frac{1}{2}} \longrightarrow 0.$$

The convergence of the second term in the right hand-side of (4.16) to zero, is a consequence of (4.10). Therefore,

$$\lim_{n\to+\infty}\left(\int_\Omega \left| a\left(x,u_n\right)\frac{\partial v}{\partial x_i} - a\left(x,u\right)\frac{\partial v}{\partial x_i}\right|^2\right)^{\frac{1}{2}} = 0$$

and this completes the proof of

$$a\left(x,u_n\right)\frac{\partial v_n}{\partial x_i} \to a\left(x,u\right)\frac{\partial v}{\partial x_i} \quad \text{in } L^2(\Omega). \tag{4.17}$$

Weak-strong convergence:

Note that from (4.14) we easily get

$$\frac{\partial u_n}{\partial x_i} \rightharpoonup \frac{\partial u}{\partial x_i} \quad \text{in } L^2(\Omega). \tag{4.18}$$

From (4.17) and (4.18) we have

$$\left\langle \frac{\partial u_n}{\partial x_i}, a\left(x,u_n\right)\frac{\partial v_n}{\partial x_i}\right\rangle \to \left\langle \frac{\partial u}{\partial x_i}, a\left(x,u\right)\frac{\partial v}{\partial x_i}\right\rangle. \tag{4.19}$$

Indeed, it is easy to see that

$$\left|\left\langle \frac{\partial u_n}{\partial x_i}, a\left(x,u_n\right)\frac{\partial v_n}{\partial x_i}\right\rangle - \left\langle \frac{\partial u}{\partial x_i}, a\left(x,u\right)\frac{\partial v}{\partial x_i}\right\rangle\right| \leq$$

$$\left|\left\langle \frac{\partial u_n}{\partial x_i}, a\left(x,u_n\right)\frac{\partial v_n}{\partial x_i} - a\left(x,u\right)\frac{\partial v}{\partial x_i}\right\rangle\right| + \left|\left\langle \frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}, a\left(x,u\right)\frac{\partial v}{\partial x_i}\right\rangle\right|, \tag{4.20}$$

then, it suffices to show that every term in the right-hand side converges to zero.

For the fist term we have

$$\left|\left\langle \frac{\partial u_n}{\partial x_i}, a\left(x,u_n\right)\frac{\partial v_n}{\partial x_i} - a\left(x,u\right)\frac{\partial v}{\partial x_i}\right\rangle\right| \leq \left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^2}\left\|a\left(x,u_n\right)\frac{\partial v_n}{\partial x_i} - a\left(x,u\right)\frac{\partial v}{\partial x_i}\right\|_{L^2}$$

and the convergence to zero is established by the fact that $\frac{\partial u_n}{\partial x_i}$ is bounded in $L^2(\Omega)$ and $a(x,u_n)\frac{\partial v_n}{\partial x_i}$ strongly converges to $a(x,u)\frac{\partial v}{\partial x_i}$ in $L^2(\Omega)$.

For the second term recall that the weak convergence of $\frac{\partial u_n}{\partial x_i}$ to $\frac{\partial u}{\partial x_i}$ in $L^2(\Omega)$, means that

$$\lim_{n\to\infty}\int_\Omega \left(\frac{\partial u_n}{\partial x_i}-\frac{\partial u}{\partial x_i}\right) g = 0, \ \ \forall g \in L^2(\Omega).$$

Then, by replacing g by $a(x,u)\frac{\partial v}{\partial x_i}$, which is in $L^2(\Omega)$, we establish the convergence of the second term in (4.20) to zero.

By summation over i, (4.19) takes the form

$$\sum_{i=1}^{N}\int_\Omega a(x,u_n)\frac{\partial u_n}{\partial x_i}\frac{\partial v_n}{\partial x_i} \longrightarrow \sum_{i=1}^{N}\int_\Omega a(x,u)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}. \tag{4.21}$$

From (4.13) we have

$$\sum_{i=1}^{N}\int_\Omega a(x,u_n)\frac{\partial u_n}{\partial x_i}\frac{\partial v_n}{\partial x_i} = \int_\Omega f v_n, \tag{4.22}$$

then, passing to the limit in (4.22), using the continuity of the linear form φ in $H_0^1(\Omega)$ and (4.21), we arrive at

$$\sum_{i=1}^{N}\int_\Omega a(x,u)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} = \int_\Omega f v, \quad \forall v \in H_0^1(\Omega).$$

Therefore, u is a solution to (4.2).

4.3 Second problem

In this section we will generalize the result obtained for problem (4.1) to:

$$\begin{cases} -\sum\limits_{i,j=1}^{N}\frac{\partial}{\partial x_i}(a_{ij}(x,u)\frac{\partial u}{\partial x_j}) = f \ \text{ in } \ \Omega \\ u = 0 \text{ in } \partial\Omega \end{cases} \tag{4.23}$$

Let Ω be a bounded domain of $\mathbb{R}^N$ with a boundary $\partial\Omega$ and let $f \in H^{-1}(\Omega)$ be given.

Theorem 3 : Suppose that $a_{ij}(.,.)$ are such that

$$a_{ij}(x,u) \in L^\infty(\Omega\times\mathbb{R}), \quad 1 \le i,j \le N$$

and satisfy the following properties:

1) $a_{ij}(.,.)$ is Charathéodory, for $1 \leq i, j \leq N$

2) there exists a positive constant α such that

$$\sum_{i,j=1}^{N} a_{ij}(x,u)\,\xi_i.\xi_j \geq \alpha\,|\xi|^2\,, \quad \forall \xi \in \mathbb{R}^N \quad \text{for a.e.} \quad x \in \Omega, \quad \forall u \in \mathbb{R}.$$

Then, problem (4.23) has a weak solution $u \in H_0^1(\Omega)$.

Weak formulation :

To obtain the weak formulation of problem (4.23), we assume that $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ is a strong solution to (4.23). By multiplying both sides of the above equation by a function $v \in C_0^\infty(\Omega)$ and integrating over Ω, we get

$$\sum_{i,j=1}^{N} \int_\Omega a_{ij}(x,u)\,\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} = \int_\Omega fv, \quad \forall v \in C_0^\infty(\Omega),$$

by using the density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$ we arrive at

$$\sum_{i,j=1}^{N} \int_\Omega a_{ij}(x,u)\,\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} = \int_\Omega fv, \quad \forall v \in H_0^1(\Omega), \tag{4.24}$$

which is the weak formulation of problem (4.23).

Thus, we a priori estimate that the solution of problem (4.24), if it exists, belongs to $H_0^1(\Omega)$.

Proof of theorem 3:

As in the first problem, we construct a family $\{V_n\}$ of finite-dimensional subspaces of $H_0^1(\Omega)$, change the given problem to a linear one, prove that this linear problem has a weak solution in each subspace V_n, then, pass to the limit using a result of the compact embedding theorem of Sobolev.

Since $H_0^1(\Omega)$ is a separable Hilbert space, so, it has an infinite Helbertian basis $\{e_m\}$. Let V_n be the finite-dimensional subspace of $H_0^1(\Omega)$ generated by $\{e_1, e_2, ..., e_n\}$.

Let $w \in V_n$ be fixed and define an approximate problem to (4.24) by

$$\sum_{i,j=1}^{N} \int_{\Omega} a_{ij}(x,w) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} = \int_{\Omega} fv. \tag{4.25}$$

It's a linear problem for which we prove the existence of a unique solution $u \in V_n$.

Let $a(.,.)$ be a bilinear form defined on V_n by

$$a(u,v) = \sum_{i,j=1}^{N} \int_{\Omega} a_{ij}(x,w) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i}$$

and let φ be a linear form defined in V_n by

$$\varphi(v) = \int_{\Omega} fv.$$

To prove that problem (4.25) has a solution in V_n, it suffices to prove that $a(.,.)$ and φ fulfill the hypotheses of the Lax-Milgram lemma.

It's easy to show that $a(.,.)$ is bilinear. We just need to check the continuity and the coercivity of $a(.,.)$.

All the norms in V_n are equivalent, so, we equip V_n by the norm induced by $H_0^1(\Omega)$ norm

$$\|u\|_{V_n} = \sum_{1 \le i \le N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2}. \tag{4.26}$$

1) **Continuity of** $a(.,.)$**:**

$$\begin{aligned} |a(u,v)| &\le \sum_{i,j=1}^{N} \int_{\Omega} |a_{ij}(x,w)| \left| \frac{\partial u}{\partial x_j} \right| \left| \frac{\partial v}{\partial x_i} \right|, \quad \forall u,v \in V_n, \\ &\le M \sum_{i,j=1}^{N} \int_{\Omega} \left| \frac{\partial u}{\partial x_j} \right| \left| \frac{\partial v}{\partial x_i} \right|, \quad \forall u,v \in V_n, \end{aligned}$$

where $M = \max_{1 \le i,j \le N} \|a_{ij}(.,w)\|_{L^\infty(\Omega)}$.

By using Cauchy Schwarz inequality we get

$$\begin{aligned}
|a(u,v)| &\leq M\sum_{i,j=1}^{N}\left(\int_{\Omega}\left|\frac{\partial u}{\partial x_j}\right|^2\right)^{\frac{1}{2}}\left(\int_{\Omega}\left|\frac{\partial v}{\partial x_i}\right|^2\right)^{\frac{1}{2}}, \quad \forall u,v\in V_n,\\
&\leq M\sum_{i,j=1}^{N}\left(\int_{\Omega}|\nabla u|^2\right)^{\frac{1}{2}}\left(\int_{\Omega}|\nabla v|^2\right)^{\frac{1}{2}}, \quad \forall u,v\in V_n.
\end{aligned}$$

Since Ω is bounded $\left(\int_{\Omega}|\nabla u|^2\right)^{\frac{1}{2}}$ defines an equivalent norm to (4.26) and

$$|a(u,v)| \leq MN^2\,\|u\|_{H_0^1}\,\|v\|_{H_0^1}\,, \quad \forall u,v\in V_n,$$

then, there exists a positive constant $C = MN^2$ such that

$$|a(u,v)| \leq C\,\|u\|_{V_n}\,\|v\|_{V_n}\,, \quad \forall u,v\in V_n.$$

Thus, $a(.,.)$ is continuous.

2) **Coercivity**:

$$\begin{aligned}
a(u,u) &= \sum_{i,j=1}^{N}\int_{\Omega}|a_{ij}(x,w)|\left|\frac{\partial u}{\partial x_j}\right|\left|\frac{\partial u}{\partial x_i}\right|\\
&\geq \alpha\int_{\Omega}|\nabla u|^2, \quad \forall u\in V_n. \qquad (4.27)
\end{aligned}$$

$\int_{\Omega}|\nabla u|^2$ defines an equivalent norm in $H_0^1(\Omega)$, consequently in V_n, so, (4.27) takes the form

$$a(u,u) \geq \alpha\,\|u\|_{V_n}^2\,, \quad \forall u\in V_n. \qquad (4.28)$$

Thus, $a(.,.)$ is coercive.

3) **Continuity of** φ :

$$|\varphi(v)| \leq \|f\|_{H^{-1}}\,\|v\|_{V_n}\,, \quad \forall v\in V_n, \qquad (4.29)$$

then, by setting $C' = \|f\|_{H^{-1}}$, we get

$$|\varphi(v)| \leq C' \|v\|_{V_n}, \quad \forall v \in V_n.$$

Thus, φ is continuous.

By using the Lax-Milgram lemma, problem (4.25) has a unique solution $u_w \in V_n$.

Furthermore, from (4.28) we have

$$\alpha \|u_w\|_{V_n}^2 \leq a(u_w, u_w)$$

and from (4.29) we have

$$\begin{aligned} a(u_w, u_w) &= \varphi(u_w) \\ &\leq \|f\|_{H^{-1}} \|u_w\|_{V_n}. \end{aligned}$$

Thus, for the solution u_w, we have the estimate

$$\|u_w\|_{V_n} \leq \frac{\|f\|_{H^{-1}}}{\alpha}. \tag{4.30}$$

Let T be the map defined on V_n by

$$T : w \longrightarrow u_w,$$

then, provided we choose w such that

$$\|w\|_{V_n} \leq \frac{\|f\|_{H^{-1}}}{\alpha},$$

$u_w = T(w)$ will be in the ball $B\left(0, \frac{\|f\|_{H^{-1}}}{\alpha}\right)$ and we can apply Brouwer fixed point theorem, provided we can prove that T is continuous.

Let $\{w_p\}$ be a convergent sequence to w in V_n and let u_p be the solution of (4.25) associated to w_p

$$\sum_{i,j=1}^{N} \int_\Omega a_{ij}(x, w_p) \frac{\partial u_p}{\partial x_j} \frac{\partial v}{\partial x_i} = \int_\Omega fv, \quad \forall v \in V_n. \tag{4.31}$$

From (4.30) the sequence $\{u_p\}$ is bounded in V_n which is a finite-dimensional space. Then, we can extract from $\{u_p\}$ a convergent subsequence $\{u_{p_k}\}$, where we denote its limit by $u \in V_n$,

$$\lim_{k\to\infty} u_{p_k} = u \ \text{ in } V_n. \tag{4.32}$$

The subsequence $\{w_{p_k}\}$ converges to w in $V_n \subset H_0^1(\Omega)$, consequently in $L^2(\Omega)$, then, we can extract again a subsequence, still denoted $\{w_{p_k}\}$, such that

$$w_{p_k} \to w, \ \text{a.e. in } \Omega.$$

Therefore, there exists $\Omega' \subset \Omega$ such that

$$meas\left(\Omega/\Omega'\right) = 0$$

and

$$w_{p_k}(x) \to w(x), \quad \forall x \in \Omega'. \tag{4.33}$$

In the other hand, since $a_{ij}(x,.)$ is continuous for a.e. x in Ω, there exists $\Omega'' \subset \Omega'$ such that

$$a_{ij}(x, w_{p_k}(x)) \to a_{ij}(x, w(x)), \quad \forall x \in \Omega'', \tag{4.34}$$

where $meas\left(\Omega'/\Omega''\right) = 0$.

Thus, from (4.33) and (4.34), we can easily show that

$$a_{ij}(x, w_{p_k}(x)) \to a_{ij}(x, w(x)) \ \text{a.e. in } \Omega, \tag{4.35}$$

consequently, for every $v \in V_n$ and $1 \leq i \leq N$, we get

$$a_{ij}(x, w_{p_k}(x)) \frac{\partial v}{\partial x_i} \to a_{ij}(x, w(x)) \frac{\partial v}{\partial x_i} \ \text{ a.e. in } \Omega.$$

Furthermore,

$$\left| a_{ij}(x, w_{p_k}(x)) \frac{\partial v}{\partial x_i} \right| \leq M \left| \frac{\partial v}{\partial x_i} \right|$$

and

$$M\left|\frac{\partial v}{\partial x_i}\right| \in L^2(\Omega).$$

Thus, by dominated convergence theorem apply to $\left\{a_{ij}\left(.,w_{p_k}\right)\frac{\partial v}{\partial x_i}\right\}$, $a_{ij}\left(.,w\right)\frac{\partial v}{\partial x_i}$ and $M\left|\frac{\partial v}{\partial x_i}\right|$, we have

$$a_{ij}\left(.,w_{p_k}\right)\frac{\partial v}{\partial x_i} \to a_{ij}\left(.,w\right)\frac{\partial v}{\partial x_i} \quad \text{in } L^2(\Omega), \quad 1 \le i,j \le N. \tag{4.36}$$

By summation over i we get

$$\sum_{i=1}^{N} a_{ij}\left(.,w_{p_k}\right)\frac{\partial v}{\partial x_i} \to \sum_{i=1}^{N} a_{ij}\left(.,w\right)\frac{\partial v}{\partial x_i} \quad \text{in } L^2(\Omega), \quad 1 \le j \le N. \tag{4.37}$$

Also, from (4.32) we have

$$\frac{\partial u_{p_k}}{\partial x_j} \to \frac{\partial u}{\partial x_j} \text{ in } L^2\left(\Omega\right), \quad 1 \le j \le N. \tag{4.38}$$

Thus, from (4.37) and (4.38) we get

$$\frac{\partial u_{p_k}}{\partial x_j}\sum_{i=1}^{N} a_{ij}\left(.,w_{p_k}\right)\frac{\partial v}{\partial x_i} \to \frac{\partial u}{\partial x_j}\sum_{i=1}^{N} a_{ij}\left(.,w\right)\frac{\partial v}{\partial x_i} \text{ in } L^1(\Omega), \quad 1 \le j \le N,$$

by summation over j we arrive at

$$\sum_{1 \le i,j \le N} a_{ij}\left(x,w_{p_k}\right)\frac{\partial u_{p_k}}{\partial x_j}\frac{\partial v}{\partial x_i} \to \sum_{1 \le i,j \le N} a_{ij}\left(x,w\right)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} \text{ in } L^1(\Omega), \forall v \in V_n. \tag{4.39}$$

Therefore, passing to the limit in (4.39), using the fact that u_{p_k} is the solution of (4.31) corresponding to w_{p_k}, we get

$$\sum_{1 \le i,j \le N} \int_\Omega a_{ij}\left(x,w\right)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} = \int_\Omega fv, \quad \forall v \in V_n,$$

then, u is the solution of (4.25) associated to w

$$u = T(w).$$

Thus, T is continuous.

Now, we are able to use the Brouwer fixed point theorem and guarantees that T has a fixed point

$$u_n \in B\left(0, \frac{\|f\|_{H^{-1}}}{\alpha}\right) \subset V_n.$$

Thus, u_n is the solution of problem (4.25) corresponding to $w = u_n$. This fact can be written by

$$\sum_{1 \leq i,j \leq N} \int_\Omega a_{ij}\left(x, u_n\right) \frac{\partial u_n}{\partial x_j} \frac{\partial v}{\partial x_i} = \int_\Omega fv, \quad \forall v \in V_n.$$

Therefore, u_n is a solution to the nonlinear problem (4.24) in V_n.

4.4 Nonlinear problem in $H_0^1(\Omega)$

Now, after we solve problem (4.24) in V_n, one would like to show that, at the limit, the solution u_n provide us with a solution to problem (4.23). For that let $v \in H_0^1(\Omega)$ and let $\{v_n\}$ be a convergent sequence to v in $H_0^1(\Omega)$, then

$$\frac{\partial v_n}{\partial x_i} \longrightarrow \frac{\partial v}{\partial x_i} \text{ in } L^2(\Omega). \tag{4.40}$$

In the other hand, for any $n \in \mathbb{N}^*$, let u_n be a solution to (4.24) in V_n, then, from (4.30) the sequence $\{u_n\}$ is bounded in $H_0^1(\Omega)$.

Thus, by using the compact embedding of $H_0^1(\Omega)$ in $L^2(\Omega)$, we can extract from $\{u_n\}$ a subsequence $\{u_{n_k}\}$ such that

$$\begin{aligned} u_{n_k} &\rightharpoonup u \text{ in } H_0^1(\Omega), \\ u_{n_k} &\longrightarrow u \text{ in } L^2(\Omega) \end{aligned}$$

and

$$u_{n_k} \longrightarrow u \text{ a.e. in } \Omega,$$

where u is an element of $H_0^1(\Omega)$, see [3], [20] and [5].

As we did in (4.35) we can show that

$$a_{ij}(x, u_{n_k}) \to a_{ij}(x, u) \text{ a.e. in } \Omega. \tag{4.41}$$

Moreover, from (4.40) and (4.41) we get

$$a_{ij}(x, u_{n_k}) \frac{\partial v_{n_k}}{\partial x_i} \longrightarrow a_{ij}(x, u) \frac{\partial v}{\partial x_i} \text{ in } L^2(\Omega).$$

Indeed, by using Minkowski's inequality we can show that

$$\begin{aligned}\left(\int_\Omega \left| a_{ij}(x, u_{n_k}) \frac{\partial v_{n_k}}{\partial x_i} - a_{ij}(x, u) \frac{\partial v}{\partial x_i} \right|^2 \right)^{\frac{1}{2}} \leq & \left(\int_\Omega \left| a_{ij}(x, u_{n_k}) \frac{\partial v_{n_k}}{\partial x_i} - a_{ij}(x, u_{n_k}) \frac{\partial v}{\partial x_i} \right|^2 \right)^{\frac{1}{2}} \\ & + \left(\int_\Omega \left| a_{ij}(x, u_{n_k}) \frac{\partial v}{\partial x_i} - a_{ij}(x, u) \frac{\partial v}{\partial x_i} \right|^2 \right)^{\frac{1}{2}},\end{aligned}$$

the first term in the right-hand side converges to zero because

$$\int_\Omega \left| a_{ij}(x, u_{n_k}) \frac{\partial v_{n_k}}{\partial x_i} - a_{ij}(x, u_{n_k}) \frac{\partial v}{\partial x_i} \right|^2 \leq M \int_\Omega \left| \frac{\partial v_{n_k}}{\partial x_i} - \frac{\partial v}{\partial x_i} \right|^2$$

and $\dfrac{\partial v_{n_k}}{\partial x_i} \longrightarrow \dfrac{\partial v}{\partial x_i}$ in $L^2(\Omega)$.

For the convergence of the second term it suffices to replace w_{p_k} by u_{n_k} in (4.36) and taking into account the density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$.

Weak-strong convergence :

From the weak convergence of u_{n_k} to u in $H_0^1(\Omega)$ we get

$$\frac{\partial u_{n_k}}{\partial x_j} \rightharpoonup \frac{\partial u}{\partial x_j}, \quad 1 \leq j \leq N.$$

Also, we have

$$a_{ij}(x, u_{n_k}) \frac{\partial v_{n_k}}{\partial x_i} \longrightarrow a_{ij}(x, u) \frac{\partial v}{\partial x_i} \text{ in } L^2(\Omega), \quad 1 \leq i \leq N, \tag{4.42}$$

then,

$$\left\langle \frac{\partial u_{n_k}}{\partial x_j}, a_{ij}(x, u_{n_k}) \frac{\partial v_{n_k}}{\partial x_i} \right\rangle \longrightarrow \left\langle \frac{\partial u}{\partial x_j}, a_{ij}(x, u) \frac{\partial v}{\partial x_i} \right\rangle, \quad 1 \leq i, j \leq N. \tag{4.43}$$

Indeed,

$$\left|\left\langle \frac{\partial u_{n_k}}{\partial x_j}, a_{ij}\left(x, u_{n_k}\right) \frac{\partial v_{n_k}}{\partial x_i} \right\rangle - \left\langle \frac{\partial u}{\partial x_j}, a_{ij}\left(x, u\right) \frac{\partial v}{\partial x_i} \right\rangle\right| \leq$$

$$\left|\left\langle \frac{\partial u_{n_k}}{\partial x_j}, a_{ij}\left(x, u_{n_k}\right) \frac{\partial v_{n_k}}{\partial x_i} - a_{ij}\left(x, u\right) \frac{\partial v}{\partial x_i} \right\rangle\right| + \left|\left\langle \frac{\partial u_{n_k}}{\partial x_j} - \frac{\partial u}{\partial x_j}, a_{ij}\left(x, u\right) \frac{\partial v}{\partial x_i} \right\rangle\right|,$$

we just need to check that each term in the right-hand side converges to zero.

For the first term we use the fact that $\{u_{n_k}\}$ is bounded in $H_0^1(\Omega)$ to get

$$\left\| \frac{\partial u_{n_k}}{\partial x_j} \right\|_{L^2} \leq \frac{\|f\|_{H^{-1}}}{\alpha},$$

also, we have

$$\begin{aligned} \left|\left\langle \frac{\partial u_{n_k}}{\partial x_j}, a_{ij}\left(x, u_{n_k}\right) \frac{\partial v_{n_k}}{\partial x_i} - a_{ij}\left(x, u\right) \frac{\partial v}{\partial x_i} \right\rangle\right| &\leq \left\| \frac{\partial u_{n_k}}{\partial x_j} \right\|_{L^2} \left\| a_{ij}\left(x, u_{n_k}\right) \frac{\partial v_{n_k}}{\partial x_i} - a_{ij}\left(x, u\right) \frac{\partial v}{\partial x_i} \right\|_{L^2}, \\ &\leq \frac{\|f\|_{H^{-1}}}{\alpha} \left\| a_{ij}\left(x, u_{n_k}\right) \frac{\partial v_{n_k}}{\partial x_i} - a_{ij}\left(x, u\right) \frac{\partial v}{\partial x_i} \right\|_{L^2}, \end{aligned}$$

so, by (4.42) we conclude that

$$\left|\left\langle \frac{\partial u_{n_k}}{\partial x_j}, a_{ij}\left(x, u_{n_k}\right) \frac{\partial v_{n_k}}{\partial x_i} - a_{ij}\left(x, u\right) \frac{\partial v}{\partial x_i} \right\rangle\right| \longrightarrow 0, \quad 1 \leq i, j \leq N.$$

The convergence of

$$\left|\left\langle \frac{\partial u_{n_k}}{\partial x_j} - \frac{\partial u}{\partial x_j}, a_{ij}\left(x, u\right) \frac{\partial v}{\partial x_i} \right\rangle\right|$$

to zero results from the weak convergence of $\frac{\partial u_{n_k}}{\partial x_j}$ to $\frac{\partial u}{\partial x_j}$, which completes the proof of (4.43).

By summation over i and j, (4.43) takes the form

$$\sum_{1\leq i,j\leq N} \int_\Omega a_{ij}\left(x, u_{n_k}\right) \frac{\partial u_{n_k}}{\partial x_j} \frac{\partial v_{n_k}}{\partial x_i} \longrightarrow \sum_{1\leq i,j\leq N} \int_\Omega a_{ij}\left(x, u\right) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i}. \tag{4.44}$$

Furthermore, since u_{n_k} is a solution to problem (4.24) in V_n we have

$$\sum_{1\leq i,j\leq N} \int_\Omega a_{ij}\left(x, u_{n_k}\right) \frac{\partial u_{n_k}}{\partial x_j} \frac{\partial v_{n_k}}{\partial x_i} = \int_\Omega f v_{n_k},$$

passing to the limit in the last equality, using (4.44) and the continuity of the linear form φ in

$H_0^1(\Omega)$, we arrive at

$$\sum_{1 \leq i,j \leq N} \int_\Omega a_{ij}(x,u) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} = \int_\Omega fv, \quad \forall v \in H_0^1(\Omega).$$

This completes the proof that u is a solution to problem (4.24) in $H_0^1(\Omega)$, consequently u is a weak solution to the given problem (4.23).

Chapter 5

Nonlinear problem involving p-Laplacian operator

5.1 Introduction

In this chapter, we study a problem involving the p-Laplacian operator of the form:

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a(x,u)\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right) = f \quad \text{in } \Omega \\ u = 0 \quad \text{in } \partial\Omega \end{cases} \tag{5.1}$$

where Ω is a domain in $\mathbb{R}^N$, $\quad N \geq 2$, with a smooth boundary and $1 < p < \infty$.

In [17], Lions studied a similar problem; namely,

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right) = f \quad \text{in } \Omega. \tag{5.2}$$

By using the monotonicity method, he was proved that for every $f \in W^{-1,p'}(\Omega)$ and for the boundary condition $u = 0$, problem (5.2) has a unique solution $u \in W_0^{1,p}(\Omega)$. For the boundary condition $u = g|_{\partial\Omega}$, where $g \in L^p(\partial\Omega)$, the solution of problem (5.2) belongs to $W^{1,p}(\Omega)$.

P-Laplacian operator:

For $u \in W^{1,p}(\Omega)$, the gradient ∇u, is defined by

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_N} \right)$$

and the Euclidian norm of ∇u is

$$|\nabla u| = \left(\sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|^2 \right)^{\frac{1}{2}}.$$

In $\mathbb{R}^N$, all the norms are equivalent, thus there exists a constant $C > 0$ such that $|\nabla u|_p \leq C\,|\nabla u|$, where

$$|\nabla u|_p = \left(\sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|^p \right)^{\frac{1}{p}}.$$

For $|\nabla u|$ we have

$$\begin{aligned} \int_\Omega |\nabla u|^p &\leq C \int_\Omega |\nabla u|_p^p \\ &\leq C \left(\sum_{i=1}^{N} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^p \right) < \infty. \end{aligned}$$

Thus,

$$|\nabla u| \in L^p(\Omega).$$

Furthermore, we have

$$\begin{aligned} \int_\Omega \left| |\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \right|^{p'} &\leq \int_\Omega \left(|\nabla u|^{p-2} |\nabla u| \right)^{p'} \\ &\leq \int_\Omega |\nabla u|^{(p-1)p'} \\ &\leq \int_\Omega |\nabla u|^p < \infty. \end{aligned}$$

Consequently,

$$|\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \in L^{p'}(\Omega), \quad 1 \leq i \leq N.$$

The p-Laplatian is the operator denoted by Δ_p and defined by

$$\begin{aligned}\Delta_p u &= div\left(|\nabla u|^{p-2}\nabla u\right)\\ &= \sum_{1\leq i\leq N}\frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right),\end{aligned}$$

this operator acts from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$ via

$$\langle \Delta_p u, v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u.\nabla v, \quad \forall u, v \in W_0^{1,p}(\Omega).$$

For more properties of the p-Laplacian see [11] and [12].

Description of the operator defining the main problem:

Let $u \in W^{1,p}(\Omega)$ and define a vector Z by

$$Z = \left(\left|\frac{\partial u}{\partial x_1}\right|^{p-2}\frac{\partial u}{\partial x_1}, \left|\frac{\partial u}{\partial x_2}\right|^{p-2}\frac{\partial u}{\partial x_2}, ..., \left|\frac{\partial u}{\partial x_N}\right|^{p-2}\frac{\partial u}{\partial x_N}\right),$$

then,

$$Z \in (L^{p'}(\Omega))^N.$$

Indeed,

$$\begin{aligned}\int_\Omega\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\left|\frac{\partial u}{\partial x_i}\right|\right)^{p'} &= \int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{(p-1)p'}, \quad 1\leq i\leq N\\ &= \int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p} < \infty, \quad 1\leq i\leq N.\end{aligned}$$

Problem (5.1) takes the form

$$\left\{\begin{array}{c} -div\,(a\,(x,u)\,Z) = f \ \text{ in } \Omega \\ u = 0 \ \text{ in } \partial\Omega \end{array}\right. .$$

Let V be a real Banach space of finite-dimensional with basis $\{e_1, e_2, ..., e_m\}$ and let $(.,.)$ be the bilinear form defined on V by

$$(\xi,\eta)=\sum_{i=1}^{m}\xi_i\eta_i,\quad \forall\xi,\eta\in V,$$

where $\xi=\sum_{i=1}^{m}\xi_i e_i$ and $\eta=\sum_{i=1}^{m}\eta_i e_i$, clearly,

$$(\xi,\xi)=\sum_{i=1}^{m}\xi_i^2=|\xi|^2\,.$$

Lemma 1:

Let $P:V\to V$ be a continuous mapping and suppose that there exists a constant $\rho>0$ such that

$$(P(\xi),\xi)\geq 0,\quad \forall\xi\in V,\ \ |\xi|=\rho. \tag{5.3}$$

Then, there exists $\xi\in V$, $|\xi|\leq\rho$ such that

$$P(\xi)=0.$$

Proof :

Let $K=\{\xi,|\xi|\leq\rho\}\subset V$ and suppose that

$$P(\xi)\neq 0,\quad \forall\xi\in K,$$

then, the function defined from K into itself by

$$\xi\to-\frac{P(\xi)\,\rho}{|P(\xi)|},$$

is continuous.

By using Brouwer fixed point theorem, there exists $\xi\in K$ such that

$$-\frac{P(\xi)\,\rho}{|P(\xi)|}=\xi.$$

Therefore, $|\xi| = \rho$ and

$$(P(\xi), \xi) = -\rho |P(\xi)| < 0;$$

which is a contradiction with (5.3). See also [17].

Definitions :

Let V be a Banach space and let $A : V \to V'$ be an operator.

We say that

1) A is monotone if and only if

$$\langle A(u) - A(v), u - v \rangle \geq 0, \quad \forall u, v \in V.$$

2) A is hemicontinuous if, for all u, v, w in V, the real-valued function defined on $\mathbb{R}$ by

$$\lambda \to \langle A(u + \lambda v), w \rangle,$$

is continuous.

Note that $\langle .,. \rangle$ is the duality pairing.

Theorem 1 :

Let V be a finite-dimensional Banach space and let $A : V \to V'$ be an operator satisfying the following proprieties

A is hemicontinuous and

$$\frac{\langle A(u), u \rangle}{\|u\|_V} \to \infty \text{ as } \|u\|_V \to \infty.$$

Then,

$$\forall f \in V', \exists u \in V \text{ such that } A(u) = f.$$

u is a weak solution.

Proof :

Let P be the mapping in V defined by

$$P(u) = \sum_{i=1}^{m} \langle A(u) - f, e_i \rangle e_i,$$

where $\{e_1, e_2, ..., e_m\}$ is a basis for V. For $u = \sum_{i=1}^{m} \xi_i e_i \in V,$ we have

$$(\xi_1, \xi_2, ..., \xi_m) \longrightarrow (\langle A(u) - f, e_1 \rangle, \langle A(u) - f, e_2 \rangle, ..., \langle A(u) - f, e_m \rangle).$$

We will show that P satisfies the property of **Lemma 1**. Thanks to the hemicontinuity of A, all the functions

$$\xi_i \to \langle A(\xi_i e_i + v_i), e_j \rangle - \langle f, e_j \rangle, \quad 1 \leq i, j \leq m$$

are continuous, where

$$v_i = \sum_{k=1, k \neq i}^{m} \xi_k e_k.$$

Thus, P is continuous.

Furthermore,

$$\begin{aligned} (P(u), u) &= \sum_{i=1}^{m} \langle A(u), e_i \rangle \xi_i - \sum_{i=1}^{m} \langle f, e_i \rangle \xi_i \\ &= \langle A(u), u \rangle - \langle f, u \rangle. \end{aligned}$$

Since $\frac{\langle A(u), u \rangle}{\|u\|_V} \to \infty$, as $\|u\|_V \to \infty$, then, $\forall \alpha > 0$ there exists $\rho > 0$ such that

$$\frac{\langle A(u), u \rangle}{\|u\|_V} \geq \alpha, \quad \forall u \in V, \quad \|u\|_V \geq \rho,$$

hence,

$$\langle A(u), u \rangle \geq \alpha \|u\|_V, \quad \forall u \in V, \quad \|u\|_V \geq \rho.$$

Let α be chosen such that $\alpha \geq \|f\|_{V'}$, then, there exists $\rho > 0$, such that

$$\langle A(u), u \rangle - \alpha \|u\|_V \geq 0, \quad \forall u \in V, \quad \|u\|_V \geq \rho,$$

which implies that

$$\langle A(u), u \rangle - \|f\|_{V'} \|u\|_V \geq 0, \text{ for } \|u\|_V \geq \rho.$$

By virtue of the inequality $\langle f, u\rangle \leq \|f\|_{V'} \|u\|_V$, we establish that

$$\langle A(u), u\rangle - \langle f, u\rangle \geq \langle A(u), u\rangle - \|f\|_{V'} \|u\|_V \geq 0, \text{ for all } u \in V, \ \ \|u\|_V \geq \rho.$$

So, by using **Lemma 1**, there exists $u \in V$ solution to the problem

$$A(u) = f.$$

5.2 The main Problem

Consider the problem

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a(x,u)\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right) = f \quad \text{in } \Omega \\ u = 0 \quad \text{in } \partial\Omega \end{cases}, \tag{5.4}$$

where $a(.,.)$ is a function satisfying the following properties

1) $a(.,.)$ is Carathéodory.

2) There exists two constants α and M such that

$$0 < \alpha \leq a(x,u) \leq M, \text{ for a.e. } x \in \Omega, \quad \forall u \in \mathbb{R}.$$

Let $w \in W_0^{1,p}(\Omega)$ be fixed and change problem (5.1) to

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a(x,w)\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right) = f \quad \text{in } \Omega \\ u = 0 \quad \text{in } \partial\Omega \end{cases}. \tag{5.5}$$

It is a linearization to problem (5.4).

Define an operator A_1 from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$ by

$$A_1(u) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a(x,w)\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right).$$

Theorem 2 : The operator A_1 is

1) bounded and hemicontinuous.

2) monotone.

3) coercive, i.e.

$$\frac{\langle A_1(u), u\rangle}{\|u\|_V} \to \infty, \text{as } \|u\|_V \to \infty.$$

Moreover, for each $f \in W^{-1,p'}(\Omega)$, $\exists u \in W_0^{1,p}(\Omega)$ such that

$$A_1(u) = f.$$

Also, if

$$\langle A_1(u) - A_1(v), u - v\rangle > 0, \ \forall u, v \in W_0^{1,p}(\Omega), \quad u \neq v$$

then, u is unique.

Proof :

Let $V = W_0^{1,p}(\Omega)$, with a dual $V' = W^{-1,p'}(\Omega)$. V and V' are reflexive and separable Banach spaces. $W_0^{1,p}(\Omega)$ has a basis $\{e_1, e_2, ..., e_m, ...\}$.

Let V_m be the finite-dimensional subspace of V generated by $\{e_1, e_2, ..., e_m\}$; that is,

$$V_m = span\,[e_1, e_2, ..., e_m],$$

then, for $u_m \in V_m$, we have

$$u_m = \sum_{i=1}^{m} \xi_i e_i.$$

1) Boundedness :

Let $S = \left\{u \in W_0^{1,p}(\Omega), \ \|u\| \leq C\right\}$. To prove that A_1 is bounded, it suffices to prove that $\{A_1(u), u \in S\}$ is bounded in $W^{-1,p'}(\Omega)$.

Indeed,

$$\|A_1(u)\|_{V'} = \sup_{v \in V, \|v\|=1} |\langle A_1(u), v\rangle|,$$

for the right-hand side, we have

$$\begin{aligned}
|\langle A_1(u), v\rangle| &= \left|\sum_{i=1}^{N}\int_{\Omega}\frac{\partial}{\partial x_i}\left(a(x,w)\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right)v dx\right| \\
&= \left|\sum_{i=1}^{N}\int_{\Omega} a(x,w)\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx\right| \\
&\leq M\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial u}{\partial x_i}\right|^{p-1}\left|\frac{\partial v}{\partial x_i}\right|dx,
\end{aligned}$$

By Hölder's inequality, we get

$$\begin{aligned}
|\langle A_1(u), v\rangle| &\leq M\sum_{i=1}^{N}\left(\int_{\Omega}\left|\frac{\partial u}{\partial x_i}\right|^{(p-1)p'}\right)^{\frac{1}{p'}}\left(\int_{\Omega}\left|\frac{\partial v}{\partial x_i}\right|^{p}\right)^{\frac{1}{p}} \\
&\leq M\sum_{i=1}^{N}\left(\int_{\Omega}\left|\frac{\partial u}{\partial x_i}\right|^{p}\right)^{\frac{1}{p'}}\left(\int_{\Omega}\left|\frac{\partial v}{\partial x_i}\right|^{p}dx\right)^{\frac{1}{p}} \\
&\leq MN\left\|\nabla u\right\|_{L^p}^{\frac{p}{p'}}\left\|\nabla v\right\|_{L^p}. \qquad (5.6)
\end{aligned}$$

Since Ω is bounded $\int_{\Omega}|\nabla u|^p$ defines an equivalent norm in $W_0^{1,p}(\Omega)$; then, (5.6) becomes

$$|\langle A_1(u), v\rangle| \leq MN\left\|u\right\|_V^{\frac{p}{p'}}\left\|v\right\|_V.$$

Also, for $u \in S$, we have

$$|\langle A_1(u), v\rangle| \leq NMC^{\frac{p}{p'}}\left\|v\right\|_V.$$

Then,

$$\sup_{v\in V_m, \|v\|_V=1}|\langle A_1(u), v\rangle| \leq NMC^{\frac{p}{p'}} = C'.$$

Thus, A_1 is bounded.

2) **Hemicontinuity :**

Set $g(t) = \langle A_1(u+tv_1), v_2\rangle$, then,

$$g(t) = \int_{\Omega} a(x,w)\left|\frac{\partial u}{\partial x_i}+t\frac{\partial v_1}{\partial x_i}\right|^{p-2}\left(\frac{\partial u}{\partial x_i}+t\frac{\partial v_1}{\partial x_i}\right)\frac{\partial v_2}{\partial x_i}.$$

To prove that A_1 is hemicontinuous, we suppose that $\{t_n\}$ is a convergent sequence to t_0 in $\mathbb{R}$ and show that

$$g(t_n) \to g(t_0).$$

For this, let h_n and h_0 be defined by

$$\begin{aligned} h_n(x) &= a(x,w)\left|\frac{\partial u}{\partial x_i} + t_n\frac{\partial v_1}{\partial x_i}\right|^{p-2}\left(\frac{\partial u}{\partial x_i} + t_n\frac{\partial v_1}{\partial x_i}\right)\frac{\partial v_2}{\partial x_i}, \quad n \in \mathbb{N}^*, \\ h_0(x) &= a(x,w)\left|\frac{\partial u}{\partial x_i} + t_0\frac{\partial v_1}{\partial x_i}\right|^{p-2}\left(\frac{\partial u}{\partial x_i} + t_0\frac{\partial v_1}{\partial x_i}\right)\frac{\partial v_2}{\partial x_i} \end{aligned}$$

and prove that h_n converges to h_0 in $L^1(\Omega)$.

Recall that the convergence of the sequence $\{t_n\}$, implies that there exists a positive constant B such that

$$|t_n| \le B, \quad \forall n \in \mathbb{N}^*.$$

Using this fact and that $a(x,w)$ is bounded by M, we get

$$|h_n(x)| \le M\left|\left|\frac{\partial u}{\partial x_i}\right| + B\left|\frac{\partial v_1}{\partial x_i}\right|\right|^{p-1}\left|\frac{\partial v_2}{\partial x_i}\right|, \tag{5.7}$$

since $\left|\frac{\partial u}{\partial x_i}\right|$ and $\left|\frac{\partial v_1}{\partial x_i}\right|$ are in $L^p(\Omega)$, then,

$$\left|\frac{\partial u}{\partial x_i}\right| + B\left|\frac{\partial v_1}{\partial x_i}\right| \in L^p(\Omega),$$

consequently,

$$\left|\left|\frac{\partial u}{\partial x_i}\right| + B\left|\frac{\partial v_1}{\partial x_i}\right|\right|^{p-1} \in L^{p'}(\Omega),$$

also, $\left|\frac{\partial v_2}{\partial x_i}\right| \in L^p(\Omega)$, therefore,

$$\left|\left|\frac{\partial u}{\partial x_i}\right| + B\left|\frac{\partial v_1}{\partial x_i}\right|\right|^{p-1}\left|\frac{\partial v_2}{\partial x_i}\right| \in L^1(\Omega). \tag{5.8}$$

In the other hand, let φ be defined by

$$\varphi(t) = |\mu(x) + t\eta(x)|^{p-2}(\mu(x) + t\eta(x)),$$

where $\mu = \frac{\partial u}{\partial x_i}$ and $\eta = \frac{\partial v_1}{\partial x_i}$.

Thanks to the continuity of φ, for $p > 1$, we get, for fixed x in Ω,

$$\varphi\left(t_n\right) \longrightarrow \varphi(t_0),$$

then, for every $x \in \Omega$, we have

$$h_n\left(x\right) \to h_0(x). \tag{5.9}$$

From (5.7), (5.8) and (5.9) we assert that $\{h_n\}$, h_0 and $M\left|\left|\frac{\partial u}{\partial x_i}\right| + B\left|\frac{\partial v_1}{\partial x_i}\right|\right|^{p-1}\left|\frac{\partial v_2}{\partial x_i}\right|$ fulfil the hypotheses of the dominated convergence theorem in $L^1\left(\Omega\right)$.

Thus, $h_n, h_0 \in L^1\left(\Omega\right)$ and

$$h_n \to h_0 \text{ in } L^1(\Omega),$$

consequently,

$$g\left(t_n\right) \longrightarrow g(t_0).$$

Therefore, A_1 is hemicontinuous.

3) Monotonicity :

Let's first prove a lemma, which we need later.

Lemma 2 : Let a, b be two real numbers and let $q > -1$, then

$$\left(|a|^q\, a - |b|^q\, b\right)\left(a - b\right) \geq 0, \quad \forall a, b \in \mathbb{R}.$$

Moreover,

$$\left(|a|^q\, a - |b|^q\, b\right)\left(a - b\right) = 0 \text{ if and only if } a = b.$$

Proof :

1) Suppose that $b \neq 0$ and $|a| > |b|$. Divide $\left(|a|^q\, a - |b|^q\, b\right)\left(a - b\right)$ by $|b|^q\, b^2$ and set $x = \frac{a}{b}$, we get

$$\left(|a|^q\, a - |b|^q\, b\right)\left(a - b\right) = \left(|x|^q\, x - 1\right)\left(x - 1\right),$$

where $|x| > 1$.

If $x > 1$, then, since $q + 1 > 0$,

$$|x|^q x = x^{q+1} > 1$$

and we easily get the result.

If $x < -1$, then

$$|x|^q x < 0,$$

so

$$(|x|^q x - 1) < 0 \text{ and } \ (x - 1) < 0,$$

hence,

$$(|x|^q x - 1)(x - 1) > 0.$$

If $|b| > |a|$, we take $x = \frac{b}{a}$ and repeat the same proof.

2) Suppose that

$$(|x|^q x - 1)(x - 1) = 0,$$

then,

$$|x|^q x - 1 = 0 \text{ or } x - 1 = 0.$$

If $x > 0$, then $|x|^q x - 1 = 0$ implies that

$$x^{q+1} = 1.$$

If $x < 0$, then $|x|^q x < 0$ and

$$|x|^q x - 1 < -1.$$

Therefore,

$$x = 1.$$

Monotonicity:

To prove that

$$\langle A_1(u) - A_1(v), u - v\rangle = \sum_{i=1}^{N} \int_{\Omega} a(x, w) \left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i} - \left|\frac{\partial v}{\partial x_i}\right|^{p-2} \frac{\partial v}{\partial x_i} \right) \left(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i} \right)$$

is nonnegative, it suffices to use the above lemma with $q = p-2$, $a = \frac{\partial u}{\partial x_i}$ and $b = \frac{\partial v}{\partial x_i}$ to get that

$$\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i} - \left|\frac{\partial v}{\partial x_i}\right|^{p-2}\frac{\partial v}{\partial x_i}\right)\left(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}\right) \geq 0,$$

then, using the fact that $a(x,w) \geq \alpha$, we arrive at

$$\langle A_1(u) - A_1(v), u - v\rangle \geq \alpha \sum_{i=1}^{N} \int_{\Omega} \left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i} - \left|\frac{\partial v}{\partial x_i}\right|^{p-2}\frac{\partial v}{\partial x_i}\right)\left(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}\right) \geq 0.$$

This completes the proof of the monotonicity of A_1.

4) **Coercivity :**

$$\begin{aligned}
\langle A_1(u), u\rangle &= -\sum_{i=1}^{N} \int_{\Omega} \frac{\partial}{\partial x_i}\left(a(x,w)\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right) u dx \\
&= \sum_{i=1}^{N} \int_{\Omega} a(x,w)\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_i} dx \\
&= \sum_{i=1}^{N} \int_{\Omega} a(x,w)\left|\frac{\partial u}{\partial x_i}\right|^{p} dx \\
&\geq \alpha\left(\sum_{i=1}^{N} \int_{\Omega} \left|\frac{\partial u}{\partial x_i}\right|^{p} dx\right) = \alpha \|u\|_V^p, \qquad (5.10)
\end{aligned}$$

which gives

$$\frac{\langle A_1(u), u\rangle}{\|u\|_{V_m}} \geq \alpha \|u\|_V^{p-1}.$$

Therefore

$$\frac{\langle A_1(u), u\rangle}{\|u\|_V} \to \infty \text{ as } \|u\|_{V_m} \to \infty, \text{ if } p > 1.$$

5.3 The approximate problem in $W_0^{1,p}(\Omega)$

Let $\{e_1, e_2, ..., e_m, ...\}$ be a basis for $W_0^{1,p}(\Omega)$, and let $V_m = span\,[e_1, e_2, ..., e_m]$, equipped with the norm induced by the $W_0^{1,p}(\Omega)$ norm.

In V_m, which is of finite-dimension, the hemicontinuity and the coercivity properties are enough to assert the existence of a weak solution u_m to the problem

$$A_1(u) = f.$$

Notice that for this solution we have

$$\langle A_1(u_m), e_j\rangle = \langle f, e_j\rangle \text{ for all } j, \quad 1 \leq j \leq m,$$

furthermore, from (5.10) we have

$$\langle A_1(u_m), u_m\rangle \geq \alpha \|u_m\|_V^p.$$

On the other hand,

$$\begin{aligned} \langle A_1(u_m), u_m\rangle &= \langle f, u_m\rangle \\ &\leq \|f\|_{V'} \|u_m\|_V \end{aligned}$$

then,

$$\alpha \|u_m\|_V^p \leq \|f\|_{V'} \|u_m\|_V,$$

hence

$$\|u_m\|_V \leq \left(\frac{\|f\|_{V_m'}}{\alpha}\right)^{\frac{1}{p-1}}.$$

Thus, the sequence $\{u_m\}$ is bounded in V, consequently, by using the boundedness of A_1, the sequence $\{A_1(u_m)\}$ is bounded in V'.

Since V and V' are reflexive spaces, we can extract a subsequence $\{u_{m_k}\}$ such that

$$u_{m_k} \rightharpoonup u \text{ in } V \text{ and } A_1(u_{m_k}) \rightharpoonup l \text{ in } V'. \tag{5.11}$$

Passing to the limit in the equality

$$\langle A_1(u_{m_k}), e_j\rangle = \langle f, e_j\rangle$$

we get

$$\langle l, e_j \rangle = \langle f, e_j \rangle \quad \forall j \geq 1,$$

hence,

$$l = f.$$

On the other hand, we are not able to pass to the limit in the left-hand side of the equality

$$\langle A_1 (u_{m_k}), u_{m_k} \rangle = \langle f, u_{m_k} \rangle,$$

however, using the weak convergence of $\{u_{m_k}\}$, we can pass to the limit in the right-hand side, hence

$$\langle A_1 (u_{m_k}), u_{m_k} \rangle = \langle f, u_{m_k} \rangle \to \langle f, u \rangle. \tag{5.12}$$

Now, using the monotonicity, we have

$$\lim_{k \to \infty} \{\langle A_1 (u_{m_k}), u_{m_k} \rangle - \langle A_1 (u_{m_k}), v \rangle - \langle A_1 (v), u_{m_k} \rangle + \langle A_1 (v), v \rangle\} \geq 0$$

and using (5.11) and (5.12) we obtain

$$\langle f - A_1 (v), u - v \rangle \geq 0 \quad \text{for all } v \in V.$$

Let $v = u + \lambda z$, where $\lambda > 0$ and $z \in V$. Using the hemicontinuity of A_1 and passing to the limit $\lambda \to 0$, in the inequality

$$\langle f, u - v \rangle \geq \langle A_1 (v), u - v \rangle$$

we get

$$\langle f, z \rangle \geq \langle A_1 (u), z \rangle. \tag{5.13}$$

Changing z by $-z$ in the last inequality, we obtain

$$\langle A_1 (u), z \rangle \leq \langle f, z \rangle. \tag{5.14}$$

From (5.13) and (5.14) we get

$$\langle A_1(u), z\rangle = \langle f, z\rangle .$$

Thus, u is a solution to problem (5.5).

5) **Uniqueness :**

For $u, v \in V$ we have seen that

$$\begin{aligned} & \langle A_1(u) - A_1(v), u - v\rangle \\ = \quad & \sum_{i=1}^{N} \int_\Omega a(x,w) \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} - \left| \frac{\partial v}{\partial x_i} \right|^{p-2} \frac{\partial v}{\partial x_i} \right) \left(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i} \right) \\ \geq \quad & \alpha \sum_{i=1}^{N} \int_\Omega \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} - \left| \frac{\partial v}{\partial x_i} \right|^{p-2} \frac{\partial v}{\partial x_i} \right) \left(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i} \right) \geq 0. \end{aligned} \tag{5.15}$$

Suppose that u and v are two solution to problem (5.5), then

$$\langle A_1(u), z\rangle = \langle f, z\rangle = \langle A_1(v), z\rangle , \quad \forall z \in V,$$

consequently,

$$\langle A_1(u) - A_1(v), u - v\rangle = 0. \tag{5.16}$$

By using (5.15) and (5.16), we arrive at

$$\left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} - \left| \frac{\partial v}{\partial x_i} \right|^{p-2} \frac{\partial v}{\partial x_i} \right) \left(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i} \right) = 0, \quad 1 \leq i \leq N,$$

then, by **lemma 2**, we have

$$\frac{\partial u}{\partial x_i} = \frac{\partial v}{\partial x_i}, \quad 1 \leq i \leq N,$$

hence,

$$u - v = c \in W_0^{1,p}(\Omega),$$

which implies that

$$u = v \text{ in } L^p(\Omega).$$

Therefore,

$$u = v \text{ a.e. in } \Omega.$$

5.4 The main problem in $W_0^{1,p}(\Omega)$

Note that the solution u to problem (5.5) is bounded

$$\|u\|_V \leq \left(\frac{\|f\|_{V/}}{\alpha}\right)^{\frac{1}{p-1}} = C_1$$

Let T be the function defined on V by

$$T(w) = u,$$

where, u is the solution to problem (5.5) corresponding to w.

Let w be chosen such that

$$\|w\|_V \leq C_1,$$

then, by vertue of Brouwer fixed point theorem, the mapping

$$T : B(0, C_1) \to B(0, C_1)$$

has a fixed point, provided we can show its continuity.

Let $\{w_k\}$ be a convergent sequence to w in V and let $\{u_k\}$ be the sequence of solutions associated to $\{w_k\}$; i.e.

$$u_k = T(w_k),$$

$\{u_k\}$ is bounded in V, which is reflexive space , then we can extract a subsequence still denoted $\{u_k\}$ and there exists $u \in V$ such that

$$u_k \rightharpoonup u \text{ in } V.$$

The subsequence $\{w_k\}$ converges strongly to w in $W_0^{1,p}(\Omega)$, hence $w_k \to w$ in $L^p(\Omega)$, so we can extract again a subsequence $\{w_{k_l}\}$ such that, $w_{k_l}(x) \to w(x)$ almost everywhere in Ω.

Using the proprieties of $a(.,.)$ we can prove; as we showed in **problem 1**; that

$$a(x, w_{k_l}) \to a(x, w) \text{ a.e. in } \Omega$$

then, for $v_1 \in W_0^{1,p}(\Omega)$, we get

$$a(x, w_{k_l}) \left|\frac{\partial v_1}{\partial x_i}\right|^{p-2} \frac{\partial v_1}{\partial x_i} \to a(x, w) \left|\frac{\partial v_1}{\partial x_i}\right|^{p-2} \frac{\partial v_1}{\partial x_i} \text{ a.e .in } \Omega.$$

Furthermore,

$$\left|a(x, w_{k_l}) \left|\frac{\partial v_1}{\partial x_i}\right|^{p-2} \frac{\partial v_1}{\partial x_i}\right| \leq M \left|\frac{\partial v_1}{\partial x_i}\right|^{p-1} \quad \text{and} \quad \left|\frac{\partial v_1}{\partial x_i}\right|^{p-1} \in L^{p'}(\Omega).$$

Thus, the sequence $\left\{a(x, w_{k_l}) \left|\frac{\partial v_1}{\partial x_i}\right|^{p-2} \frac{\partial v_1}{\partial x_i}\right\}$, $a(x, w) \left|\frac{\partial v_1}{\partial x_i}\right|^{p-2} \frac{\partial v_1}{\partial x_i}$ and $M \left|\frac{\partial v_1}{\partial x_i}\right|^{p-1}$ fulfil the hypotheses of the dominated convergence theorem, consequently,

$$a(x, w_{k_l}) \left|\frac{\partial v_1}{\partial x_i}\right|^{p-2} \frac{\partial v_1}{\partial x_i} \to a(x, w) \left|\frac{\partial v_1}{\partial x_i}\right|^{p-2} \frac{\partial v_1}{\partial x_i} \text{ in } L^{p'}(\Omega).$$

Therefore, for $v \in W_0^{1,p}(\Omega)$, we get

$$a(x, w_{k_l}) \left|\frac{\partial v_1}{\partial x_i}\right|^{p-2} \frac{\partial v_1}{\partial x_i} \frac{\partial v}{\partial x_i} \to a(x, w) \left|\frac{\partial v_1}{\partial x_i}\right|^{p-2} \frac{\partial v_1}{\partial x_i} \frac{\partial v}{\partial x_i} \text{ in } L^1(\Omega),$$

which we can write as

$$\sum_{i=1}^{N} \int_{\Omega} a(x, w_{k_l}) \left|\frac{\partial v_1}{\partial x_i}\right|^{p-2} \frac{\partial v_1}{\partial x_i} \frac{\partial v}{\partial x_i} dx \to \sum_{i=1}^{N} \int_{\Omega} a(x, w) \left|\frac{\partial v_1}{\partial x_i}\right|^{p-2} \frac{\partial v_1}{\partial x_i} \frac{\partial v}{\partial x_i} dx. \tag{5.17}$$

If we define an operator A_k by

$$A_k(v_1) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(a(x, w_{k_l}) \left|\frac{\partial v_1}{\partial x_i}\right|^{p-2} \frac{\partial v_1}{\partial x_i} \right),$$

(5.17) shows that

$$\langle A_k(v_1), v\rangle \to \langle A_1(v_1), v\rangle . \ \forall v \in W_0^{1,p}(\Omega). \tag{5.18}$$

Also, because u_{k_l} is a solution to

$$A_k(u) = f,$$

we have

$$\langle A_k(u_{k_l}), v\rangle = \langle f, v\rangle, \ \forall v \in W_0^{1,p}(\Omega). \tag{5.19}$$

Moreover, using the weak convergence of $\{u_{k_l}\}$ to u, we obtain

$$\langle A_k(u_{k_l}), u_{k_l}\rangle = \langle f, u_{k_l}\rangle \to \langle f, u\rangle . \tag{5.20}$$

Recall that

$$a(x, w_{k_l})\left|\frac{\partial v}{\partial x_i}\right|^{p-2}\frac{\partial v}{\partial x_i} \to a(x, w)\left|\frac{\partial v}{\partial x_i}\right|^{p-2}\frac{\partial v}{\partial x_i} \text{ strongly in } L^{p'}(\Omega)$$

and

$$\frac{\partial u_{k_l}}{\partial x_i} \rightharpoonup \frac{\partial u}{\partial x_i} \text{ weakly in } L^p(\Omega),$$

then, using the weak-strong convergence as we did in problem 1, we get

$$\langle A_k(v), u_{k_l}\rangle = \sum_{i=1}^{N}\int_{\Omega} a(x, w_{k_l})\left|\frac{\partial v}{\partial x_i}\right|^{p-2}\frac{\partial v}{\partial x_i}\frac{\partial u_{k_l}}{\partial x_i}dx \to \langle A_1(v), u\rangle . \tag{5.21}$$

By using (5.18)-(5.21) and

$$\langle A_k(u_{k_l}) - A_k(v), u_{k_l} - v\rangle = \langle A_k(u_{k_l}), u_{k_l}\rangle - \langle A_k(u_{k_l}), v\rangle - \langle A_k(v), u_{k_l}\rangle + \langle A_k(v), v\rangle \geq 0$$

and passing to the limit we get

$$\langle f, u\rangle - \langle f, v\rangle - \langle A_1(v), u\rangle + \langle A_1(v), v\rangle \geq 0,$$

therefore,

$$\langle A_1(v), v - u\rangle \ \geq \langle f, v - u\rangle .$$

Replacing v in the last inequality by $u+\lambda z$, for $z \in W_0^{1,p}(\Omega)$, $\lambda > 0$ and repeating the same steps as in (5.13), (5.14), we obtain

$$\langle A_1(u), v\rangle = \langle f, v\rangle, \ \forall v \in W_0^{1,p}(\Omega).$$

Thus, u is the weak solution to (5.5) associated to w

$$u = T(w),$$

which shows that T is continuous.

Finally, by using the Brouwer fixed point theorem for T, there exists $u \in B(0, C_1)$ such that

$$u = T(u).$$

That is

$$\sum_{i=1}^{N} \int_\Omega a(x,u) \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx = \int_\Omega fv, \quad \forall v \in W_0^{1,p}(\Omega).$$

Therefore, u is a weak solution to problem (5.4).

Chapter 6

Second problem involving p-Laplacian operator

6.1 Introduction

In this chapter, we consider the problem

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(a(x,u) \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) + b(x) u = f \quad \text{in } \Omega \\ u = 0 \quad \text{in } \partial\Omega \end{cases} \tag{6.1}$$

where Ω is a domain in $\mathbb{R}^N$ with a smooth boundary.

Let A be the operator defined from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$ by

$$A(u) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(a(x,u) \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) + b(x) u.$$

Remark : Recall that a function $u \in W_0^{1,p}(\Omega)$ satisfying, for a given $f \in W^{-1,p'}(\Omega)$,

$$\langle A(u), v \rangle = \langle f, v \rangle, \quad \forall v \in W_0^{1,p}(\Omega),$$

is called a weak solution to problem (6.1).

We state some theorems which serve us later.

Theorem 1: (**Existence**)

Let V be a finite-dimensional Banach space and let $A : V \to V'$ be an operator satisfying the following proprieties

1) A is hemicontinuous.

2) $\frac{\langle A(u),u\rangle}{\|u\|_V} \to \infty$ as $\|u\|_V \to \infty$.

Then

$$\forall f \in V', \ \exists u \in V \ \text{ such that } A(u) = f.$$

Theorem 2:(Rillich Kondrachov)

Let Ω be an open bounded subset of $\mathbb{R}^N$ with a C^1 boundary. Suppose that $1 \leq p < N$ and let $p^* = \frac{Np}{N-p}$. Then, for each $1 \leq q < p^*$ the embedding

$$W^{1,p}(\Omega) \subset L^q(\Omega)$$

is compact. Moreover we have the estimate

$$\|u\|_{L^q(\Omega)} \leq \beta \|u\|_{W^{1,p}(\Omega)}. \tag{6.2}$$

for some β depending only on p and N.

Also, by using (6.2) and the equivalence between the norms $\|u\|_{W_0^{1,p}(\Omega)} = \|\nabla u\|_{L^p(\Omega)}$ and $\|u\|_{W^{1,p}(\Omega)}$ in $W_0^{1,p}(\Omega)$ we get, for $u \in W_0^{1,p}(\Omega)$,

$$\|u\|_{L^q(\Omega)} \leq \beta' \|\nabla u\|_{L^p(\Omega)}, \tag{6.3}$$

where β' is a positive constant depending only on p, N and Ω.

6.2 An approximate problem

For a fixed $w \in W_0^{1,p}(\Omega)$, let A_1 be the operator defined by

$$A_1(u) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a(x,w)\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right) + b(x)\,u \tag{6.4}$$

Properties of A_1 in $W_0^{1,p}(\Omega)$:

Suppose that $a(.,.) \in L^\infty(\Omega \times \mathbb{R})$ and $b \in L^\infty(\Omega)$.

Furthermore, suppose that a is Charathéodory and there exist two constants α and M such that

$$0 < \alpha \leq a(x,u) \leq M, \text{ for almost every } x \in \Omega, \quad \forall u \in \mathbb{R}$$

and

$$0 < \alpha \leq b(x) \leq M, \text{ for almost every } x \in \Omega.$$

Proposition 3:

Let $V = W_0^{1,p}(\Omega)$ and $A_1 : V \to V'$ be the operator defined by (6.4), then for p satisfying

$$\frac{2N}{2+N} \leq p < N, \quad N \geq 3$$

a) The operator A_1 is

1) bounded and hemicontinuous.

2) monotone.

3) coercive; i.e.

$$\frac{\langle A_1(u), u\rangle}{\|u\|_V} \to \infty, \text{ as } \|u\|_V \to \infty.$$

b) for each $f \in W^{-1,p'}(\Omega)$, $\exists u \in W_0^{1,p}(\Omega)$ such that

$$A_1(u) = f. \tag{6.5}$$

Proof: 1) **Boundedness :**

Set $S = \left\{u \in W_0^{1,p}(\Omega),\ \|u\|_{W_0^{1,p}} \leq C\right\}$, then $A(S) = \{A_1(u),\ u \in S\}$ is bounded in $W^{-1.p'}(\Omega)$.

Indeed,

for every v in $W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}
|(A_1(u), v)| &= \left| -\sum_{i=1}^{N} \int_{\Omega} \frac{\partial}{\partial x_i} \left(a(x,w) \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) v dx + \int_{\Omega} b(x) uv dx \right| \\
&\leq \left| \sum_{i=1}^{N} \int_{\Omega} a(x,w) \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx \right| + \left| \int_{\Omega} b(x) uv dx \right| \\
&\leq M \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{p-1} \left| \frac{\partial v}{\partial x_i} \right| dx + M \int_{\Omega} |uv| \, dx. \qquad (6.6)
\end{aligned}$$

In order to be able to use Hölder's inequality in each integral in the last inequality, we need to choose u and v in $W_0^{1,p}(\Omega) \cap L^2(\Omega)$. So it suffices to choose p such that

$$W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega).$$

From **Theorem 2,** p must satisfies the inequalities

$$\begin{aligned}
1 &\leq p < N, \\
p^* &= \frac{Np}{N-p} \geq 2,
\end{aligned}$$

therefore,

$$\frac{1}{p} - \frac{1}{N} \leq \frac{1}{2},$$

then,

$$\frac{2N}{2+N} \leq p < N.$$

Now, we are able to use Hölder's inequality for (6.6) to get

$$\begin{aligned}
|(A_1(u), v)| &\leq M \sum_{i=1}^{N} \left(\int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{(p-1)p'} \right)^{\frac{1}{p'}} \left(\int_{\Omega} \left| \frac{\partial v}{\partial x_i} \right|^{p} \right)^{\frac{1}{p}} + M \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\
&\leq MN \left(\int_{\Omega} |\nabla u|^p \right)^{\frac{1}{p'}} \left(\int_{\Omega} |\nabla v|^p \right)^{\frac{1}{p}} + M \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}
\end{aligned}$$

which we can also write

$$|(A_1(u), v)| \leq MN \|\nabla u\|_{L^p(\Omega)}^{\frac{p}{p'}} \|\nabla v\|_{L^p(\Omega)} + M \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}. \tag{6.7}$$

Since $u \in S$ then $\|\nabla u\|_{L^p(\Omega)} \leq C$, consequently, we get from (6.3) and (6.7)

$$\begin{aligned} |(A_1(u), v)| &\leq MNC^{\frac{p}{p'}} \|\nabla v\|_{L^p(\Omega)} + M(\beta')^2 C \|\nabla v\|_{L^p(\Omega)} \\ |(A_1(u), v)| &\leq \delta \|v\|_{W_0^{1,p}(\Omega)}, \end{aligned}$$

where $\delta = M \max\left\{NC^{\frac{p}{p'}},\ (\beta')^2 C\right\}$, so,

$$\|A_1(u)\|_{V'} = \sup_{v \in V, \|v\|=1} |\langle A_1(u), v\rangle| \leq \delta.$$

Therefore, A_1 is bounded.

2) **Hemicontinuity**:

Set $g(t) = \langle A_1(u + tv_1), v_2\rangle$; that is

$$g(t) = \sum_{i=1}^{N} \int_\Omega a(x, w) \left|\frac{\partial u}{\partial x_i} + t\frac{\partial v_1}{\partial x_i}\right|^{p-2} \left(\frac{\partial u}{\partial x_i} + t\frac{\partial v_1}{\partial x_i}\right) \frac{\partial v_2}{\partial x_i} + \int_\Omega b(x)(u + tv_1) v_2.$$

Suppose that $\{t_n\}$ is a convergent sequence to t_0 in $\mathbb{R}$. We will show that

$$g(t_n) \to g(t_0).$$

Let (h_n), (k_n), h_0 and k_0 be defined by

$$\begin{aligned} h_n(x) &= a(x, w) \left|\frac{\partial u(x)}{\partial x_i} + t_n\frac{\partial v_1(x)}{\partial x_i}\right|^{p-2} \left(\frac{\partial u(x)}{\partial x_i} + t_n\frac{\partial v_1(x)}{\partial x_i}\right) \frac{\partial v_2(x)}{\partial x_i}, \text{ for } n \in \mathbb{N}^* \\ h_0(x) &= a(x, w) \left|\frac{\partial u(x)}{\partial x_i} + t_0\frac{\partial v_1(x)}{\partial x_i}\right|^{p-2} \left(\frac{\partial u(x)}{\partial x_i} + t_0\frac{\partial v_1(x)}{\partial x_i}\right) \frac{\partial v_2(x)}{\partial x_i} \end{aligned}$$

and

$$\begin{aligned} k_n(x) &= b(x)(u(x) + t_n v_1(x)) v_2(x), \quad \text{for } n \in \mathbb{N}^* \\ k_0(x) &= b(x)(u(x) + t_0 v_1(x)) v_2(x). \end{aligned}$$

We have already proved in **Theorem 2 of the previous** chapter, that

$$h_n \to h_0 \text{ in } L^1(\Omega). \tag{6.8}$$

Let us prove that

$$k_n \to k_0 \text{ in } L^1(\Omega).$$

Set $\psi(t) = b(x)(u(x) + t_n v_1(x)) v_2(x)$ for fixed x in Ω, clearly ψ is continuous, consequently,

$$k_n(x) \to k_0(x) \text{ every where in } \Omega.$$

Also, since $\{t_n\}$ is a convergent sequence, there exists a constant $B > 0$ such that

$$|t_n| \leq B, \ \forall n \in N^*,$$

then, by using the boundedness of b and t_n, we get

$$\begin{aligned} \int_\Omega |k_n(x)| \, dx &\leq M \int_\Omega (|u| + B|v_1|) |v_2| \, dx \\ &\leq M \, \| |u| + B|v_1| \|_{L^2(\Omega)} \|v_2\|_{L^2(\Omega)} \leq \infty, \end{aligned}$$

therefore,

$$k_n \in L^1(\Omega), \quad \forall n \in N^*$$

and there exists a function

$$k = M(|u| + B|v_1|)|v_2| \in L^1(\Omega)$$

such that

$$|k_n(x)| \leq k(x) \text{ a.e. in } \Omega.$$

Thus, the sequence $\{k_n(x)\}$, $k_0(x)$ and $k(x)$ satisfy the hypotheses of the dominated convergence theorem, hence

$$k_0 \in L^1(\Omega) \text{ and } k_n \to k_0 \text{ in } L^1(\Omega). \tag{6.9}$$

From (6.8) and (6.9) we conclude that

$$h_n + k_n \to h_0 + k_0 \text{ in } L^1(\Omega)$$

which means that

$$g(t_n) \to g(t_0),$$

therefore, A_1 is hemicontinuous.

3) **Monotonicity :**

We have

$$\begin{aligned}
\langle A_1(u) - A_1(v), u - v\rangle &= \sum_{i=1}^{N} \int_{\Omega} a(x,w) \left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i} - \left|\frac{\partial v}{\partial x_i}\right|^{p-2} \frac{\partial v}{\partial x_i} \right) \left(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i} \right) dx \\
&\quad + \int_{\Omega} b(x)(u-v)^2 \\
&\geq \sum_{i=1}^{N} \alpha \int_{\Omega} \left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i} - \left|\frac{\partial v}{\partial x_i}\right|^{p-2} \frac{\partial v}{\partial x_i} \right) \left(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i} \right) dx \\
&\quad + \alpha \int_{\Omega} |u-v|^2 \, dx,
\end{aligned}$$

since $\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i} - \left|\frac{\partial v}{\partial x_i}\right|^{p-2} \frac{\partial v}{\partial x_i} \right) \left(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i} \right) \geq 0$, as we have proved in **Lemma 1** of the previous chapter, $\alpha > 0$ and $\|u - v\|^2_{L^2(\Omega)} \geq 0$, then

$$\langle A_1(u) - A_1(v), u - v\rangle \geq 0.$$

Therefore, A_1 is monotone.

4) **Coercivity :**

By using the properties of $a(.,.)$ and b, we easily show that

$$\begin{aligned}
\langle A_1(u), u\rangle &= \sum_{i=1}^{N} \int_{\Omega} a(x,w) \left|\frac{\partial u}{\partial x_i}\right|^p dx + \int_{\Omega} b(x)\, u^2(x)\, dx \\
&\geq \alpha \left(\sum_{i=1}^{N} \int_{\Omega} \left|\frac{\partial u}{\partial x_i}\right|^p dx + \int_{\Omega} u^2(x)\, dx \right) \\
&\geq \alpha \left(\|u\|_{W_0^{1,p}}^p + \|u\|_{L^2(\Omega)}^2 \right) \\
&\geq \alpha \|u\|_{W_0^{1,p}}^p , \qquad (6.10)
\end{aligned}$$

which gives

$$\frac{\langle A_1(u), u\rangle}{\|u\|_{W_0^{1,p}}} \to \infty \text{ as } \|u\|_{W_0^{1,p}} \to +\infty, \text{ if } p > 1.$$

Therefore, A_1 is coercive.

6.3 The approximate problem in finite-dimensional space

Let $\{e_1, e_2, ..., e_m, ...\}$ be a basis for $W_0^{1,p}(\Omega)$ and let $V_m = span\,[e_1, e_2, ..., e_m]$, so,

$$u_m = \sum_{i=1}^{m} \xi_i e_i, \quad \forall u_m \in V_m.$$

We equip V_m with the norm induced by $W_0^{1,p}(\Omega)$ norm.

By using **Theorem 1** for V_m, which is of finite-dimension, the problem $A_1(u) = f$ has a weak solution u_m.

This solution satisfies

$$\langle A_1(u_m), e_j\rangle = \langle f, e_j\rangle \text{ for all } j, \ 1 \leq j \leq m.$$

Also, from (6.10) we have

$$\langle A_1(u_m), u_m\rangle \geq \alpha \|u_m\|_{W_0^{1,p}}^p .$$

On the other hand, we have

$$\langle A_1(u_m), u_m\rangle = \langle f, u_m\rangle,$$

consequently,

$$\begin{aligned}\alpha \|u_m\|^p_{W_0^{1,p}} &\leq \langle f, u_m\rangle \\ &\leq \|f\|_{W^{-1,p'}} \|u_m\|_{W_0^{1,p}}.\end{aligned}$$

Thus,

$$\|u_m\|_{W_0^{1,p}} \leq \left(\frac{\|f\|_{W^{-1,p'}}}{\alpha}\right)^{\frac{1}{p-1}}, \tag{6.11}$$

therefore, the sequence $\{u_m\}$ is bounded in $W_0^{1,p}(\Omega)$. As a consequence of the boundedness property of A_1, $\{A_1(u_m)\}$ is bounded in $W^{-1,p'}(\Omega)$.

Because $W_0^{1,p}$ and $W^{-1,p'}$ are reflexive spaces, we can extract a subsequence $\{u_{m_k}\}$ from $\{u_m\}$, such that

$$u_{m_k} \rightharpoonup u \text{ in } W_0^{1,p}(\Omega) \text{ and } A_1(u_{m_k}) \rightharpoonup l \text{ in } W^{-1,p'}(\Omega), \tag{6.12}$$

thus, using the weak convergence of $\{u_{m_k}\}$ and $\{A_1(u_{m_k})\}$, we arrive at

$$\langle A_1(v), u_{m_k}\rangle \to \langle A_1(v), u\rangle, \quad \forall v \in W_0^{1,p}(\Omega)$$

and

$$\langle A_1(u_{m_k}), v\rangle \to \langle l, v\rangle, \quad \forall v \in W_0^{1,p}(\Omega). \tag{6.13}$$

Recall that u_{m_k} is a weak solution to problem (6.5), then

$$\langle A_1(u_{m_k}), e_j\rangle = \langle f, e_j\rangle.$$

Replacing v by e_j in (6.13) and passing to the limit in the last equality, we arrive at

$$\langle l, e_j\rangle = \langle f, e_j\rangle, \ \forall j \geq 1,$$

which implies that

$$f = l. \tag{6.14}$$

Also, using the weak convergence of $\{u_{m_k}\}$, the right-hand side in the equality

$$\langle A_1(u_{m_k}), u_{m_k}\rangle = \langle f, u_{m_k}\rangle$$

converges to $\langle f, u\rangle$, so

$$\langle A_1(u_{m_k}), u_{m_k}\rangle = \langle f, u_{m_k}\rangle \to \langle f, u\rangle. \tag{6.15}$$

Thus, we are able to pass to the limit in each term in the inequality

$$\langle A_1(u_{m_k}), u_{m_k}\rangle - \langle A_1(u_{m_k}), v\rangle - \langle A_1(v), u_{m_k}\rangle + \langle A_1(v), v\rangle \geq 0,$$

which; by using (6.12), (6.14) and (6.15); gives

$$\langle f, u\rangle - \langle f, v\rangle - \langle A_1(v), u\rangle + \langle A_1(v), v\rangle \geq 0.$$

Thus,

$$\langle f - A_1(v), u - v\rangle \geq 0, \quad \forall v \in W_0^{1,p}(\Omega),$$

therefore,

$$\langle f, u - v\rangle \geq \langle A_1(v), u - v\rangle, \quad \forall v \in W_0^{1,p}(\Omega). \tag{6.16}$$

Let $\lambda > 0$ and take $v = u + \lambda z$ in (6.16) we get

$$\langle f, z\rangle \geq \langle A_1(u + \lambda z), z\rangle, \quad \forall z \in W_0^{1,p}(\Omega).$$

Passing to the limit for $\lambda \to 0$ in the last inequality and using the hemicontinuity of A_1, we arrive at

$$\langle f, z\rangle \geq \langle A_1(u), z\rangle, \quad \forall z \in W_0^{1,p}(\Omega), \tag{6.17}$$

changing z by $-z$ in (6.17) we obtain

$$\langle f, z\rangle \leq \langle A_1(u), z\rangle, \quad \forall z \in W_0^{1,p}(\Omega). \tag{6.18}$$

From (6.17) and (6.18) we get

$$\langle f, z\rangle = \langle A_1(u), z\rangle, \quad \forall z \in W_0^{1,p}(\Omega),$$

which means that

$$A_1(u) = f.$$

6.4 The problem in $W_0^{1,p}(\Omega)$

Let T be the function $T : w \to u$, where u is the weak solution of problem (6.5) associated to w.

From (6.11) we can assert that the solution u is bounded,

$$\|u\|_{W_0^{1,p}} \leq \left(\frac{\|f\|_{W^{-1,p}}}{\alpha}\right)^{\frac{1}{p-1}} = C_1.$$

If we choose w such that $\|w\|_{W_0^{1,p}} \leq C_1$, then the mapping

$$T : B(0, C_1) \to B(0, C_1)$$

is continuous.

To prove the continuity of T, we consider a convergent sequence $\{w_k\}$ to w in $W_0^{1,p}(\Omega)$ and prove that

$$T(w_k) \to T(w).$$

For that let $\{u_k\}$ be the sequence of weak solutions to (6.5) associated to $\{w_k\}$, i.e.

$$\sum_{i=1}^{N} \int_{\Omega} a(x, w_k) \left|\frac{\partial u_k}{\partial x_i}\right|^{p-2} \frac{\partial u_k}{\partial x_i} \frac{\partial v}{\partial x_i} + \int_{\Omega} b(x) u_k v = \int_{\Omega} f v, \quad \forall v \in W_0^{1,p}(\Omega) \tag{6.19}$$

and define an operator A_k by

$$A_k(u) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a(x, w_k) \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) + b(x)\, u(x),$$

then, (6.19) takes the form

$$\langle A_k(u_k), v \rangle = \langle f, v \rangle, \quad \forall v \in W_0^{1,p}(\Omega). \tag{6.20}$$

Recall that $\{u_k\}$ is bounded in $W_0^{1,p}(\Omega)$, which is a reflexive space, then we can extract a subsequence still denoted $\{u_k\}$ and there exists $u \in W_0^{1,p}(\Omega)$ such that

$$u_k \rightharpoonup u \text{ in } W_0^{1,p}(\Omega).$$

By using the compact embedding of $W_0^{1,p}(\Omega)$ in $L^2(\Omega)$, we get

$$u_k \to u \text{ in } L^2(\Omega).$$

In the previous chapter we showed that for every $v, v_1 \in W_0^{1,p}(\Omega)$ we have

$$\sum_{i=1}^{N} \int_\Omega a(x, w_k) \left| \frac{\partial v}{\partial x_i} \right|^{p-2} \frac{\partial v}{\partial x_i} \frac{\partial v_1}{\partial x_i} dx \to \sum_{i=1}^{N} \int_\Omega a(x, w) \left| \frac{\partial v}{\partial x_i} \right|^{p-2} \frac{\partial v}{\partial x_i} \frac{\partial v_1}{\partial x_i} dx.$$

Therefore, since $v, v_1 \in W_0^{1,p}(\Omega) \subset L^2(\Omega)$ and $|bvv_1| \leq M\, |vv_1|$, the integral $\int_\Omega b(x)\, v(x)\, v_1(x)\, dx$ makes sense and

$$\sum_{i=1}^{N} \int_\Omega a(x, w_k) \left| \frac{\partial v}{\partial x_i} \right|^{p-2} \frac{\partial v}{\partial x_i} \frac{\partial v_1}{\partial x_i} dx + \int_\Omega b(x)\, v(x)\, v_1(x)\, dx$$

converges to

$$\sum_{i=1}^{N} \int_\Omega a(x, w) \left| \frac{\partial v}{\partial x_i} \right|^{p-2} \frac{\partial v}{\partial x_i} \frac{\partial v_1}{\partial x_i} dx + \int_\Omega b(x)\, v(x)\, v_1(x)\, dx$$

which means that

$$\langle A_k(v), v_1 \rangle \to \langle A_1(v), v_1 \rangle, \qquad \forall v, v_1 \in W_0^{1,p}(\Omega),$$

in particular, for $v_1 = v$ we have

$$\langle A_k(v), v\rangle \to \langle A_1(v), v\rangle, \quad \forall v \in W_0^{1,p}(\Omega). \tag{6.21}$$

Moreover, using (6.20) and the weak convergence of $\{u_k\}$ to u in $W_0^{1,p}(\Omega)$ we obtain

$$\langle A_k(u_k), u_k\rangle = \langle f, u_k\rangle \to \langle f, u\rangle. \tag{6.22}$$

We have already seen that

$$a(x, w_k)\left|\frac{\partial v}{\partial x_i}\right|^{p-2}\frac{\partial v}{\partial x_i} \to a(x, w)\left|\frac{\partial v}{\partial x_i}\right|^{p-2}\frac{\partial v}{\partial x_i} \text{ strongly in } L^{p'}(\Omega)$$

and

$$\frac{\partial u_k}{\partial x_i} \rightharpoonup \frac{\partial u}{\partial x_i} \text{ weakly in } L^p(\Omega),$$

then, using the weak-strong convergence as we have used in problem 1, we get

$$\sum_{i=1}^{N}\int_{\Omega} a(x, w_k)\left|\frac{\partial v}{\partial x_i}\right|^{p-2}\frac{\partial v}{\partial x_i}\frac{\partial u_k}{\partial x_i}dx \to \sum_{i=1}^{N}\int_{\Omega} a(x, w)\left|\frac{\partial v}{\partial x_i}\right|^{p-2}\frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i}dx \tag{6.23}$$

Moreover, for $v \in L^2(\Omega)$, we have $|bv| \le M|v|$, then

$$bv \in L^2(\Omega),$$

since $u_k \to u$ in $L^2(\Omega)$, then

$$u_k bv \to ubv \text{ in } L^1(\Omega),$$

which yields

$$\int_{\Omega} b(x)\, u_k(x)\, v(x)\, dx \to \int_{\Omega} b(x)\, u(x)\, v(x)\, dx. \tag{6.24}$$

From (6.23) and (6.24) we get

$$\langle A_k(v), u_k\rangle \to \langle A_1(v), u\rangle. \tag{6.25}$$

By using (6.22), (6.20), (6.25) and (6.21), we are able to pass to the limit in each term in the inequality

$$\langle A_k(u_k) - A_k(v), u_k - v\rangle = \langle A_k(u_k), u_k\rangle - \langle A_k(u_k), v\rangle - \langle A_k(v), u_k\rangle + \langle A_k(v), v\rangle \geq 0,$$

to get

$$\langle f, u\rangle - \langle f, v\rangle - \langle A_1(v), u\rangle + \langle A_1(v), v\rangle \geq 0.$$

Therefore,

$$\langle A_1(v), v - u\rangle \geq \langle f, v - u\rangle.$$

Replacing v by $u + \lambda z$ for $\lambda > 0$ and repeating the same work as in (6.17) and (6.18), we obtain

$$\langle A_1(u), z\rangle = \langle f, z\rangle, \quad \forall z \in W_0^{1,p}(\Omega).$$

Thus, u is the weak solution to the problem

$$A_1(u) = f,$$

therefore,

$$u = T(w)$$

which completes the proof that T is continuous.

Since T is continuous from $B(0, C_1)$ into itself, the Brouwer fixed point theorem guarantees the existence of $u \in B(0, C_1)$ such that

$$u = T(u).$$

Thus, problem (6.5) corresponding to $w = u$, takes the form

$$\sum_{i=1}^{N} \int_\Omega a(x, u) \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} + \int_\Omega b(x)\, uv = \int_\Omega fv, \quad \forall v \in W_0^{1,p}(\Omega),$$

which implies that

$$A(u) = f.$$

Therefore, problem (6.1) has u as a weak solution.

Chapter 7

Maximum principle

7.1 Introduction

The maximum principle asserts that solutions of certain elliptic equations of second order cannot have a maximum or a minimum in the interior of the domain of definition [19]. The basic idea is quite simple, if a solution u, to an elliptic equation, has a maximum at a point x and the second derivatives of u do not all vanish at x, then the matrix $\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)$ must be negative definite at x, in contradiction to the equation.

However, maximum principle can be used to show that solution to certain equations must be nonnegative. This is important for quantities which have physical interpretation as densities and concentrations [21].

The aim of this chapter is to formulate a maximum principle for solutions of nonlinear elliptic equations of the form

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(a(x,u) \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) = f \quad \text{in } \Omega \tag{7.1}$$

and

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(a(x,u) \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) + b(x)\, u = f \quad \text{in } \Omega.$$

We start with a proposition which serves us later.

Proposition 1: Let Ω be an open set of $\mathbb{R}^N$ and $1 \le p < \infty$. Suppose that $h \in W_0^{1,p}(\Omega)$ and $\nabla h = 0$ in Ω, then

$$h = 0 \text{ in } \Omega.$$

Proof :

We extend h to $\mathbb{R}^N$ by

$$\tilde{h}(x) = \begin{cases} h(x) \text{ in } \Omega \\ 0 \text{ in } \mathbb{R}^N/\Omega \end{cases}$$

then,

$$\tilde{h} \in W^{1,p}\left(\mathbb{R}^N\right) \text{ and } \nabla\tilde{h} = \widetilde{\nabla h},$$

see [5].

Under the assumption made on ∇h, we have $\widetilde{\nabla h} = 0$, consequently $\tilde{h}$ is constant in $\mathbb{R}^N$ [5], since $\tilde{h} \in L^p(\mathbb{R}^N)$, then

$$\tilde{h} = 0.$$

7.2 Maximum principle for solutions to p-Laplacian problem

Now, we derive a maximum principle for the problem

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a(x,u)\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right) = f \quad \text{in } \Omega \\ u_{|\partial\Omega} = g \end{cases} \tag{7.2}$$

where $a(.,.)$ is in $L^\infty(\Omega \times \mathbb{R})$ and satisfies the property

$$\exists \alpha > 0 \text{ and } \beta > 0, \text{ such that } \alpha < a(x,u) < \beta \text{ for a.e. } x \in \Omega, \text{ and } u \in \mathbb{R},$$

the weak form of (7.2) is

$$\sum_{i=1}^{N} \int_\Omega a(x;u)\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx = \int_\Omega fv dx, \quad \forall v \in W_0^{1,p}(\Omega). \tag{7.3}$$

Theorem 2: Assume that $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ is a solution to (7.2) and f, g are such that

$$\begin{aligned} f &\leq 0 \text{ a.e. in } \Omega, \\ u &= g \leq 0 \text{ a.e. in } \partial\Omega, \end{aligned}$$

then,

$$u \leq 0 \text{ in } \overline{\Omega}.$$

Proof :

Let G be a $C^1(\mathbb{R})$ function satisfying the following properties

1) G is strictly increasing in $(0; +\infty)$,

2) $G(s) = 0$ for $s \leq 0$,

3) there exists $M > 0$, such that $|G'(s)| \leq M, \quad \forall s \in (0; +\infty)$.

Under the assumptions made in G, we have

$$G(u) \in W_0^{1,p}(\Omega).$$

Indeed,

$$\begin{aligned} |G(u)| &= |G(u) - G(0)| \\ &\leq |G'(\xi) u| \\ &\leq M|u|, \end{aligned}$$

then, we get

$$G(u) \in L^p(\Omega).$$

Also,

$$\frac{\partial G(u)}{\partial x_i} = G'(u) \frac{\partial u}{\partial x_i},$$

by using the third property of G, we have

$$\left| \frac{\partial G(u)}{\partial x_i} \right| \leq M \left| \frac{\partial u}{\partial x_i} \right|.$$

Since $\frac{\partial u}{\partial x_i} \in L^p(\Omega)$, we conclude that

$$\nabla G \in \left(L^p(\Omega)\right)^N .$$

By using the fact that $u \in C\left(\overline{\Omega}\right)$ and $G \in C^1(\mathbb{R})$ we get

$$G(u) \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).$$

Furthermore, since $g \leq 0$, we have $u(x) \leq 0$ for $x \in \partial\Omega$, then

$$G(u(x)) = 0, \quad \forall x \in \partial\Omega,$$

since Ω is bounded, we have

$$G(u) \in W_0^{1,p}(\Omega).$$

By choosing $v = G(u)$ in (7.3), we arrive at

$$\sum_{i=1}^{N} \int_\Omega a(x;u) \left|\frac{\partial u}{\partial x_i}\right|^p G'(u)\, dx = \int_\Omega fG(u)\, dx,$$

using the property of $a(.,.)$ and the signs of $\left|\frac{\partial u}{\partial x_i}\right|^p$ and $G'(u)$ we get

$$\sum_{i=1}^{N} \int_\Omega \alpha \left|\frac{\partial u}{\partial x_i}\right|^p G'(u)\, dx \leq \int_\Omega fG(u)\, dx,$$

therefore, since $f \leq 0$, α and G are positives, we conclude that

$$\int_\Omega |\nabla u|_p^p G'(u)\, dx \leq 0, \tag{7.4}$$

where $|.|_p$ is the p-Euclidian norm in $\mathbb{R}^N$ defined by $|x|_p^p = \sum_{i=1}^{N} |x_i|^p$.

Define H by

$$H(s) = \int_0^s \left(G'(t)\right)^{\frac{1}{p}} dt,$$

then

$$H \in C^1(\mathbb{R}).$$

By using the fact that $H'(s) = (G'(s))^{\frac{1}{p}}$, we easily get

$$H(s) = 0, \quad \forall s \leq 0,$$

and

$$H(s) > 0, \quad \forall s > 0. \tag{7.5}$$

On the other hand, we have

$$|H(u)| \leq M^{\frac{1}{p}} |u| \tag{7.6}$$

and

$$\frac{\partial H(u)}{\partial x_i} = \frac{\partial}{\partial u}\left(\int_0^u (G'(t))^{\frac{1}{p}} dt\right)\frac{\partial u}{\partial x_i} = (G'(u))^{\frac{1}{p}} \frac{\partial u}{\partial x_i}, \tag{7.7}$$

consequently,

$$\left|\frac{\partial H(u)}{\partial x_i}\right| \leq M^{\frac{1}{p}} \left|\frac{\partial u}{\partial x_i}\right|. \tag{7.8}$$

From (7.6) and (7.8) we conclude that

$$H(u) \in W^{1,p}(\Omega) \cap C\left(\overline{\Omega}\right).$$

Furthermore, $u(x) \leq 0$ in $\partial\Omega$, which implies that

$$H(u) = 0, \quad \forall x \in \partial\Omega.$$

Thus,

$$H(u) \in W_0^{1,p}(\Omega).$$

Moreover, from (7.7) we have

$$\nabla H(u) = (G'(u))^{\frac{1}{p}} \nabla u,$$

hence, using the fact that $G'(u) \geq 0$, we get

$$|\nabla H(u)|_p^p = G'(u)\,|\nabla u|_p^p\,,$$

then (7.4) becomes

$$\int_\Omega |\nabla H(u)|_p^p\,dx \leq 0.$$

Since $|\nabla H(u)|_p^p$ is nonnegative the last inequality isn't true unless that $|\nabla H(u)|_p^p = 0$.

By using the result of **Proposition 1** for H we get

$$H(u) = 0, \quad \forall x \in \Omega,$$

then, using (7.5) the last equality implies that

$$u(x) \leq 0 \text{ in } \Omega.$$

Corollary 3: Suppose that Ω is bounded, and the solution u to problem (7.2) is such that

$$u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).$$

If

$$f \leq 0 \text{ a.e. in } \Omega,$$

then

$$u(x) \leq \sup_{x\in\partial\Omega} g(x).$$

Proof: Let $K = \sup_{x\in\partial\Omega} g(x)$, then $w = u - K \in W^{1,p}(\Omega)$ (because Ω is bounded) and $\frac{\partial w}{\partial x_i} = \frac{\partial u}{\partial x_i}$, which implies that w satisfies the equation

$$\sum_{i=1}^{N} \int_\Omega a(x; w+K) \left|\frac{\partial w}{\partial x_i}\right|^{p-2} \frac{\partial w}{\partial x_i}\frac{\partial v}{\partial x_i}dx = \int_\Omega fv dx, \quad \forall v \in W_0^{1,p}(\Omega). \tag{7.9}$$

If we set $b(x,w) = a(x;w+K)$, then

$$b(x,w) > \alpha, \text{ for a.e. } x \in \Omega, \text{ and } w \in \mathbb{R};$$

consequently the equation (7.9) becomes

$$\sum_{i=1}^{N}\int_{\Omega} b\left(x;w\right)\left|\frac{\partial w}{\partial x_i}\right|^{p-2}\frac{\partial w}{\partial x_i}\frac{\partial v}{\partial x_i}dx=\int_{\Omega} fvdx \quad \forall v\in W_0^{1,p}(\Omega).$$

The last equation has the same form and the same assumptions as (7.3), therefore, by using the result of **Theorem 2** we conclude that the solution w verifies

$$w\leq 0 \text{ in } \overline{\Omega}$$

which means that

$$u\leq K \text{ in } \overline{\Omega}.$$

Remarks :

1- Notice that if Ω is of C^1-boundary, or $u\in W_0^{1,p}\left(\Omega\right)$, then it is not necessary to suppose that $u\in C(\overline{\Omega})$, because there is a possibility to assigning a boundary values along $\partial\Omega$ to a function $u\in W^{1,p}(\Omega)$.

2- If we change the assumptions made on the signs of f and g by

$$f\geq 0 \text{ in } \Omega \text{ and } g\geq 0 \text{ on } \partial\Omega,$$

then we get

$$u\geq 0 \text{ in } \overline{\Omega}.$$

7.3 A maximum principle for second p-Laplacian problem

In this section we derive a maximum principle for the following problem

$$\begin{cases} -\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(a\left(x,u\right)\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right)+b\left(x\right)u=f \quad \text{in } \Omega \\ u_{|\partial\Omega}=g \end{cases} \tag{7.10}$$

where Ω is bounded domain of $\mathbb{R}^N$, $a\left(.,.\right)$ is Charathéodory and there exist two constant α and β such that

$$0 < \alpha \leq a(x,u), b(x) \leq \beta, \text{ for almost every } x \in \Omega, \quad \forall u \in \mathbb{R}.$$

The weak form of problem (7.10) is

$$\sum_{i=1}^{N} \int_{\Omega} a(x,u) \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} + \int_{\Omega} b(x) uv = \int_{\Omega} fv, \quad \forall v \in W_0^{1,p}(\Omega). \tag{7.11}$$

Theorem 4 : Suppose that $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ is a weak solution to (7.10), then

$$u \leq \max \left\{ \sup_{\Omega} (f/b), \sup_{\partial\Omega} g \right\}.$$

Proof : We use the truncation method of **Stampacchia**.

Let G be the function defined in the proof of **Theorem 1** and let

$$K = \max \left\{ \sup_{\Omega} (f/b), \sup_{\partial\Omega} g \right\},$$

then

$$G(u-K) \in W_0^{1,p}(\Omega).$$

Replace v by $G(u-K)$ in (7.11) and subtracting $\int_{\Omega} b(x) K G(u-K)$ we get

$$\sum_{i=1}^{N} \int_{\Omega} a(x,u) \left| \frac{\partial u}{\partial x_i} \right|^{p} G'(u-K) + \int_{\Omega} b(x)(u-K) G(u-K) = \int_{\Omega} (f - b(x) K) G(u-K).$$

By using the fact that $0 < \alpha \leq a(x,u)$, taking into account that $|\nabla u|^p G'(u-K) \geq 0$ and that $b(x) \neq 0$ the last equality becomes

$$\alpha \int_{\Omega} |\nabla u|^p G'(u-K) + \int_{\Omega} b(x)(u-K) G(u-K) \leq \int_{\Omega} b(x) \left(\frac{f(x)}{b(x)} - K \right) G(u-K),$$

then,

$$\begin{aligned} \int_{\Omega} b(x)(u-K) G(u-K) \quad &\leq \quad \int_{\Omega} b(x) \left(\frac{f(x)}{b(x)} - K \right) G(u-K) \\ &\quad -\alpha \int_{\Omega} |\nabla u|^p G'(u-K). \end{aligned} \tag{7.12}$$

The last integral in right-hand side of (7.12) is nonnegative

$$\alpha \int_{\Omega} |\nabla u|^p \, G'(u-K) \geq 0,$$

for the first integral in the right-hand side we notice that $b(x) > 0$, $G(u-K) \geq 0$ and $\frac{f(x)}{b(x)} - K \leq 0$ for every $x \in \Omega$, then

$$\int_{\Omega} b(x) \left(\frac{f(x)}{b(x)} - K \right) G(u-K) \leq 0.$$

Thus,

$$\int_{\Omega} b(x) (u-K) \, G(u-K) \leq 0, \tag{7.13}$$

but $G(u-K) = 0$ if $u - K \leq 0$, then

$$\int_{\Omega} b(x) (u-K) \, G(u-K) = \int_{\Omega_+} b(x) (u-K) \, G(u-K), \tag{7.14}$$

where $\Omega_+ = \{x \in \Omega; u - K > 0\}$.

By using the fact that $b(x)(u-K) G(u-K) \geq 0$ in Ω_+ we have

$$\int_{\Omega_+} b(x) (u-K) \, G(u-K) \geq 0. \tag{7.15}$$

From (7.13), (7.14) and (7.15) we get

$$\int_{\Omega_+} b(x) (u-K) \, G(u-K) = 0,$$

which implies that

$$u - K = 0,$$

or

$$b(x) \, G(u-K) = 0.$$

Since $b(x) > 0$, the last equality implies that

$$G(u-K) = 0,$$

so, $u \leq K$, or

$$meas\left(\Omega_{+}\right)=0,$$

which means that $u \leq K$ a.e. in Ω, hence everywhere in Ω, because $u \in C(\overline{\Omega})$. Therefore,

$$u \leq \max\left\{\sup_{\Omega}\left(f/b\right), \sup_{\partial\Omega} g\right\}, \quad \forall x \in \overline{\Omega}. \tag{7.16}$$

Corolary: In addition to the assumptions of Theorem 4, suppose that u verifies

$$a\left(x,-u\right)=a\left(x,u\right), \text{ for a.e. } x \in \Omega,$$

then

$$\min\left\{\inf_{\Omega}\left(f/b\right), \inf_{\partial\Omega} g\right\} \leq u \leq \max\left\{\sup_{\Omega}\left(f/b\right), \sup_{\partial\Omega} g\right\}, \quad \forall x \in \overline{\Omega}. \tag{7.17}$$

Proof: By using the assumption made in a, (7.11) can be written

$$-\sum_{i=1}^{N}\int_{\Omega} a\left(x,-u\right)\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial\left(-u\right)}{\partial x_i}\frac{\partial v}{\partial x_i}-\int_{\Omega} b\left(x\right)\left(-u\right)v=\int_{\Omega} fv, \quad \forall v \in W_0^{1,p}(\Omega),$$

consequently,

$$\sum_{i=1}^{N}\int_{\Omega} a\left(x,-u\right)\left|\frac{\partial\left(-u\right)}{\partial x_i}\right|^{p-2}\frac{\partial\left(-u\right)}{\partial x_i}\frac{\partial v}{\partial x_i}+\int_{\Omega} b\left(x\right)\left(-u\right)v=\int_{\Omega} -fv, \quad \forall v \in W_0^{1,p}(\Omega),$$

which has the same form as (7.11) with $-u$ and $-f$ in place of u and f respectively.

Therefore, (7.16) gives

$$-u \leq \max\left\{\sup_{\Omega}\left(-f/b\right), \sup_{\partial\Omega}\left(-g\right)\right\}, \quad \forall x \in \overline{\Omega}.$$

Thus, by using the fact that

$$\begin{aligned}\sup_{\Omega}\left(-f/b\right) &= -\inf_{\Omega}\left(f/b\right),\\ \sup_{\partial\Omega}\left(-g\right) &= -\inf_{\partial\Omega} g\end{aligned}$$

and

$$\max\{-\lambda,-\delta\} = -\min\{\lambda,\delta\}$$

we get easily (7.17).

Corollary 5: If $f \geq \lambda \geq 0$ a.e. in Ω and $g \geq \gamma$ a.e. in $\partial\Omega$ then

$$u(x) \geq \min\{\lambda/\beta,\gamma\}.$$

In particular, if

$$f \geq 0 \text{ a.e. in } \Omega \text{ and } g \geq 0 \text{ a.e. in } \partial\Omega,$$

then,

$$u(x) \geq 0 \text{ in } \overline{\Omega}.$$

Proof: It suffices to show that

$$\inf_{\Omega}(f/b) \geq \lambda/\beta \text{ and } \inf_{\partial\Omega} g \geq \gamma,$$

then, from (7.17) we get

$$u(x) \geq \min\{\lambda/\beta,\gamma\}.$$

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