International Journal of Mathematical Archive-2(5), May - 2011, Page: 693-697
Available online through www.ijma.info ISSN 2229 – 5046

STUDY OF NONLINEAR ELASTICITY PROBLEM BY ELLIPTIC REGULARIZATION WITH LAMÉ SYSTEM

Meflah Mabrouk*

University of K. Merbah Ouargla, Algeria Department of Mathematics Faculty of Sciences

E-mail: meflah@ymail.com

(Received on: 06-01-11; Accepted on: 18-01-11)

ABSTRACT

In this work, we look for the existence and uniqueness of a function u = u(x, t), x ∈ Ω, t ∈ [0, T] solution of the nonlinear boundary value problem by the elliptic regularization techniques based on theory of monotone operators.

***Keywords**: Priori estimates, monotone operators, Lamé system, elliptic regularization.*

---

1. NOTATIONS AND POSITION OF THE PROBLEM:

Let Ω an open bounded domain of IR^n, with regular boundary Γ. We denote by Q the cylinder $IR^n_x \times IR_t$: Q = Ω × [0,T], with boundary Σ. L designed Lamé system define by μΔ + (λ+μ) ∇div, λ and μ are constants Lamé with λ+ μ ≥0. and h, f are functions. We look for the existence and uniqueness of a function $u = u(x, t)$, $x \in \Omega$, t ∈[0,T], solution of the problem (P):

$$(P)\begin{cases} u'' - Lu + +|u'|^{p-2}u' = f & , \quad in\ Q \quad (1.1.1)\\ u = 0 & , \quad on\ \Sigma \quad (1.1.2)\\ u(x,0) = u(x,T) & , \quad \forall\, x\in\Omega \quad (1.1.3)\\ u'(x,0) = u'(x,T) & , \quad \forall\, x\in\Omega \quad (1.1.4)\end{cases} \tag{1.1}$$

2. EXISTENCE OF THE SOLUTION:

Theorem: 1 Assume that Ω is bounded open of IR^n are given f, with f ∈ L(Q). Then there exists a function $u = w_0+w$ satisfying (P)

$$w_0 \in H_0^1\ (\Omega) + W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \tag{1.2}$$

$$w \in\ L^2\big(0,T; H_0^1\ (\Omega)\big) \tag{1.3}$$

$$w \in\ L^p(Q) \tag{1.4}$$

Proof: we use an approach due to G. Prodi [11] we have

$$\begin{cases} u = w_0 + w\\ w_0\ independent\ of\ T\\ \int_0^T w dt = 0\end{cases} \tag{1.5}$$

We introduce the Prodi idea (1. 5) in (1.1.1) we having

$$u'' - Lu + |u'|^{p-2}u' - f = f + Lu_0 \tag{1.6}$$

We consider the derivative of (1.6) we obtain

$$\frac{d}{dt}(u'' - Lu + |u'|^{p-2}u') = \frac{df}{dt} \tag{1.7}$$

Corresponding author: Meflah Mabrouk, *E-mail: meflah@ymail.com
University of K. Merbah Ouargla, Algeria Department of Mathematics Faculty of Sciences

And

$$\begin{cases} \int_0^T u dt = 0 \\ u(T) = u(0) \\ u'(x,0) = u'(x,T) \end{cases} \tag{1.8}$$

We deduce to (1.7)

$$u'' - Lu + |u'|^{p-2}u' - f = h_0 \ \ with\ h_0\ independent\ of\ t \tag{1.9}$$

For resolve (1.7) and (1.8) we denotes. L = -A; $\beta(u') = |u'|^{p-2}u'$

And we define the functional space V:

$$V = \begin{cases} v: v \in L^2(0,T,H_0^1(\Omega)); \quad v' \in L^2(0,T,(H_0^1(\Omega)) \cap L^p(Q); \\ v'' \in L^2(0,T,L^2(\Omega)); \ \int_0^T v(t)dt = 0; \ \ v(T) = v(0); \ v'(T) = v'(0) \end{cases} \tag{1.10}$$

The Banach structure of V is defined by

$$\|v\|_V = \|v\|_{L^2(0,T,H_0^1(\Omega))} + \|v'\|_{L^2(0,T,H_0^1(\Omega))} + \|v\|_{L^p(Q)} + \|v\|_{L^2(0,T,L^2(\Omega))}$$

We define the bilinear form:

$$b(u,v) = \int_0^T [(u'',v) + (Au,v) + (\beta(u'),v)]dt \tag{1.11}$$

The weak formulation of (1.7) and (1.8) is to find $u \epsilon V$ such that

$$b(u,v) = \int_0^T (f,v')dt \quad \forall v \epsilon V \tag{1.12}$$

Following some ideas of Lions for obtain the elliptic regularization, given $\delta > 0$ and $u, v\ \epsilon V$ we define

$$\pi_\delta(u,v) = \delta \int_0^T [(u'',v'') + (Au',v')]ds + \int_0^T (u'' + Au + \beta(u'),v')ds. \tag{1.13}$$

The application $v \to \pi_\delta(u,v)$ is continuous on V so there existes an application $\pi_\delta\ \epsilon\ V'$:

$$\pi_\delta(u,v) = (B_\delta\ (u),v) \tag{1.14}$$

The linear operator $B_\delta : V \to V'$ satisfies the four properties:

B_δ is bounded in V' for all bounded set in V and is a hemi continuous and is a strictly monotonous and is coercive. In view of these properties and as consequence of theorem of Lions [5], there exist unique a function $u_\delta \epsilon\ V$:

$$\pi_\delta(u_\delta,v) = \int_0^T (f,v')dt \quad \forall\ v \epsilon V \tag{1.15}$$

2.1 A PRIORI ESTIMATES I:

Explicitly the elliptic regularization (1.15) and setting $v = u_\delta$, we obtain:

$$\delta \int_0^T [|u''_\delta|^2 + \|u'_\delta\|^2]dt + \int_0^T [|u'_\delta|^2 + (\beta(u'_\delta),u'_\delta)]dt = \int_0^T (f,u_\delta)dt \tag{1.16}$$

Or

$\int_0^T (\beta(u'),u')dt = \|u'\|^p_{L^p(Q)}$ and $\int_0^T u dt = 0 \Rightarrow \|u\|_{L^2(0,T,H_0^1(\Omega))} \leq C\|u'\|_{L^2(0,T,H_0^1(\Omega))}$

Then

$$u'_\delta \text{is bounded in } L^p(Q) \text{ when } \delta \to 0 \tag{1.17}$$

$$\delta \int_0^T [|u''_\delta|^2 + |u'_\delta|^2 + \|u'_\delta\|^2]dt \leq C \tag{1.18}$$

Or

$\int_0^T u_\delta dt = 0,$ we have by (1.17) and (1.18) that:

u_δ is bounded in $L^p(Q)$ (1.19)

$$\delta \int_0^T \|u_\delta\|^2 dt \le C \quad (1.20)$$

2.2 A PRIORI ESTIMATES II:

Exchange in (1.15) v with:

$$v(t) = \int_0^T u_\delta(s)d - \frac{1}{T}\int_0^T (T-s)u_\delta(s)ds \quad (1.21)$$

We verify that:

$$\begin{cases} \int_0^T v dt = 0 \quad \forall v \in V \\ \quad v' = u_\delta \end{cases} \quad (1.22)$$

Taking into account (1.21) in (1.15) we get

$$\delta \int_0^T [(u''_\delta, u'_\delta) + (u'_\delta, u_\delta) + (Au'_\delta, u_\delta)]dt + \int_0^T [(u''_\delta, u_\delta) + (u'_\delta, u_\delta) + \|u_\delta\|^2]dt$$
$$+ \int_0^T (\beta(u'_\delta), u'_\delta)dt = \int_0^T (f, u_\delta)dt \quad (1.23)$$

By using periodicity of $u_\delta, u'_\delta \epsilon\, V$ we obtain:

$$\int_0^T (u''_\delta, u'_\delta)dt = \int_0^T (Au'_\delta, u_\delta)dt = 0 \quad (1.24)$$

And

$$\int_0^T (u''_\delta, u_\delta)dt = (u'_\delta(T), u_\delta(T)) - (u'_\delta(0), u_\delta(0)) - \int_0^T (u'_\delta, u'_\delta)dt = -\int_0^T |u'_\delta|^2 dt \quad (1.25)$$

By (1.24) and (1.17) we have

$$\left|\int_0^T (u''_\delta, u_\delta)dt\right| \le C \quad when \quad \delta \to 0 \quad (1.26)$$

Also, from (1.17) and (1.19) we obtain:

$$\left|\int_0^T (\beta(u'_\delta), u_\delta)dt\right| \le \|\beta(u'_\delta)\|_{L^p(Q)} \|u_\delta\|_{L^p(Q)} \le C' \quad (1.27)$$

Combining (1.24), (1.26), (1.27) with (1.23) we deduce

$$\int_0^T \|u_\delta\|^2 dt \le C \quad (1.28)$$

2.3 PASSAGE TO THE LIMIT:

From (1.17) and (1.28) that there exists a subsequence from (u_δ), such that

$$u_\delta \to 0 \quad weak\ in \quad L^2(0, T, H_0^1(\Omega)) \quad (1.29)$$

$$u'_\delta \to u' \quad weak\ in \quad L^p(Q) \quad (1.30)$$

$$\beta(u'_\delta) \to \chi \ weak\ in \quad L^q(Q) \quad (1.31)$$

Passage to the limit in (1.15) we obtain

$$\int_0^T [(-u', v'') + (Au, v') + (\chi, v')]dt = \int_0^T (f, v')dt \quad \forall v \in V \quad (1.32)$$

Use the convolution technical in (1.32) we have

$$\int_0^T (\chi, u' * \eta_\delta * \eta_\delta)dt = \int_0^T (f, u' * \eta_\delta * \eta_\delta)dt \quad \forall v \in V \tag{1.33}$$

When

$$\int_0^T (\chi, u'')dt = \int_0^T (f, u')dt \qquad \forall v \in V \tag{1.34}$$

3. UNIQUENESS OF SOLUTION:

Theorem: Under the hypotheses of the theorem of existence, we consider two solutions u_1 and u_2 of the problem (P) then $u_1 = u_2$.

Proof: We subtract the equations (1.5) corresponding to u_1 and u_2 and sitting $\emptyset = u_1 - u_2$ we have:

$$\emptyset'' + \emptyset' + A\emptyset + \beta(u_1') - \beta(u_2') \tag{2.1}$$

Denoting by (η_δ) the regularizing sequence a similar argument by Brézis [2] we obtain

$$\emptyset' * \eta_\delta * \eta_\delta = \emptyset * \eta'_\delta * \eta_\delta \tag{2.2}$$

Hence, by using (1.2) and (1.3), we have

$$\emptyset = \varphi + \emptyset_0 : \quad \emptyset_0 \in V \; and \; \varphi L^2(0, T, H_0^1(\Omega)) \tag{2.3}$$

From (2.2) we get

$$\emptyset' * \eta_\delta * \eta_\delta = \emptyset * \eta'_\delta * \eta_\delta = \varphi' * \eta_\delta * \eta_\delta \tag{2.4}$$

Show that

$$\int_0^T (\emptyset'', \emptyset' * \eta_\delta * \eta_\delta)dt = 0 .$$

When

$$\int_0^T \frac{d}{dt}(\emptyset', \emptyset' * \eta_\delta * \eta_\delta)dt = \int_0^T (\emptyset'', \emptyset' * \eta_\delta * \eta_\delta)dt + \int_0^T (\emptyset', \emptyset'' * \eta_\delta * \eta_\delta)dt = 2\int_0^T (\emptyset'', \emptyset' * \eta_\delta * \eta_\delta)dt = 0 \tag{2.5}$$

Therefore

$$\int_0^T (\emptyset'', \emptyset' * \eta_\delta * \eta_\delta)dt = \frac{1}{2}\int_0^T \frac{d}{dt}(\emptyset', \emptyset' * \eta_\delta * \eta_\delta)dt = 0 \tag{2.6}$$

$\emptyset'$ and η_δ Periodic then we have

$$\int_0^T (\emptyset, \emptyset' * \eta_\delta * \eta_\delta)dt = \int_0^T (\emptyset', \emptyset' * \eta_\delta * \eta_\delta)dt = \int_0^T (A\emptyset, \emptyset' * \eta_\delta * \eta_\delta)dt = 0 \tag{2.7}$$

From (2.1); (2.6) and (2.7) we obtain:

$$\int_0^T (\beta(u'_1) - \beta(u'_2), \emptyset' * \eta_\delta * \eta_\delta) = 0 \tag{2.8}$$

Passage to the limit in (2.8) we have

$$\int_0^T (\beta(u'_1) - \beta(u'_2), u'_1 - u'_2)dt = 0 \tag{2.9}$$

Where

$$u'_1 - u'_2 = 0 \Rightarrow u'_1 = u'_2 \tag{2.10}$$

This implies that

$$\emptyset = u_1 - u_1 = \theta, \; \theta \text{ independent of t} \tag{2.11}$$

From (2.7) *and* (2.11) *we obtain*

$\int_0^T (A\theta, \theta) dt = 0 \quad \forall\theta \in V$ (2.12)

We deduce from (1.2)

$\theta \in H_0^1\ (\Omega) + W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ (2.13)

By (2.12) and (2.13) and using theorem of Green we have (Aθ,θ) = 0⇒ θ = 0.

where the uniqueness of solution.

ACKNOWLEDGEMENT: The author would like his thanks to the Prof. B. Merouani and LAMA Ferhat Abass University.

REFERENCES:

[1] R.A. Adams, Sobolev Spaces, Academic Press, (1976).

[2] H. Brezis, Analyse fonctionnelle, théorie et applications. Masson (1983).

[3] H. Brezis, Equations et inéquations non linéaires dans les espaces vectoriels en dualité, Ann, Ins.Fourier; 18, (1968), 115-175.

[4] G.Duvaut, J.L. Lions, Les inéquations en mécanique et en physique. Dunod. Paris. (1972).

[5] J.L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod. (1969).

[6] Luc Tartar, Topics in non lineair analysis. Université de Paris-Sud, Publications Mathematiques d'Orsay, novembre (1978).

[7] M. Meflah, A Nonlinear Elasticity Problem by Elliptic Regularization Technics, Int. J. Contemp. Math. Sciences, Vol 6, 2011, no. 25, 1221-1229.

[8] M. Meflah, B Merouani, A Nonlinear Elasticity Problem Governed by Lamé System, Applied Mathematical Sciences, Vol.4, 2010, no. 36, 1785-1796.

[9] B. Merouani, M. Meflah, A. Boulaouad, The Generalized and perturbed Lamé system, Applied Mathematical SciencesVol.2, 2008,no. 49,2425-2430.

[10] F. Messelmi, B. Merouani, M. Meflah, Nonlinear thermoelasticity problem, Analele Universitatii Oradea, Fasc. Matematica,Tom XV (2008), 207-217.

[11] G. Prodi, Soluzioni periodiche dell'equazione delle onde com termine dissipativo nonlineare.Rend. Sem.Mat. Pandova, 35 (1965).

[12] V. Patron, P. Perline, Méthode de la théorie mathématique de l'élasticité, Editions Mir, Moscou, 1981.

[13] Taylor M.E, Partial differential non linear equations III Springer. (1996).

[14] K. Yosida, Functional Analysis, Springer-Verlag Berlin Heidelberg New York 1968.

****

