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A Nonlinear Elasticity Problem by Elliptic Regularization Technics

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Abstract

In this work, we look for the existence and uniqueness of a function $u = u(x, t)$, $x \in \Omega$, $t \in (0, T)$ solution of the nonlinear boundary value problem by the elliptic regularization technics.

1 Notations and position of the problem

Let Ω an open bounded domain of $\mathbb{R}^n$, with regular boundary Γ.We denot by Q the cylinder $\mathbb{R}^n_x \times \mathbb{R}_t$: $Q = \Omega \times \,]0, T[$, with boundary Σ. L designe Lamé system define by $\mu\Delta + (\lambda + \mu)\nabla div$; λ and μ are constants Lamé with $\lambda +\ \mu \geq 0$.and f, h are functions, $\rho = p - 2$.We look for the existence and uniqueness of a function $u = u(x, t)$, $x \in \Omega$, $t \in]0, T[$, solution of the problem $(P) : (1; 2; 3; 4)$.

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - Lu + \left|\frac{\partial u}{\partial x}\right|^{\rho} \frac{\partial u}{\partial x} = f \quad \text{in } Q \tag{1}$$

$$u = 0 \qquad\qquad \text{on } \Sigma \tag{2}$$

$$u(x, 0) = u(x, T) \qquad x \in \Omega \tag{3}$$

$$\frac{\partial u}{\partial t}(x, 0) = \frac{\partial u}{\partial t}(x, T) \qquad x \in \Omega \tag{4}$$

2 Existence and uniqueness of the solution

2.1 Existence of the solution

The technics we use are those of the method by elliptic regularization

Theorem 1 *Assume that* Ω *is bounded open of* $\mathbb{R}^n$ *are given* f*,with* $f \in L^q(Q)$ *.Then there exists a function* $u = w_0 + w$ *satisfying* $(1, 2$ *and* $3)$

$$w_0 \in H_0^1(\Omega) + W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \tag{5}$$

$$w \in L^2\left(0, T; H_0^1(\Omega)\right), \tag{6}$$

$$w' \in L^p(Q) \tag{7}$$

2.1.1 First step: looking for approached solutions

we use an approach due to G.Prodi [10] we have

$$\begin{cases} u = w_0 + w \\ w_0 \text{ independent of } T \\ \displaystyle\int_0^T w dt = 0 \end{cases} \tag{8}$$

We introduce the Prodi idia (8) in (1) we having the following properties:

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - Lu + \left|\frac{\partial u}{\partial x}\right|^\rho \frac{\partial u}{\partial x} = f + Lu_0 \tag{9}$$

We consider the derivative of (9) we obtain

$$\frac{d}{dt}\left(\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - Lu + \left|\frac{\partial u}{\partial x}\right|^\rho \frac{\partial u}{\partial x}\right) = \frac{df}{dt}. \tag{10}$$

and

$$\int_0^T u dt = 0; \qquad u(T) = u(0); \qquad \frac{\partial u}{\partial t}(x, 0) = \frac{\partial u}{\partial t}(x, T) \tag{11}$$

We deduce to (10)

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - Lu + \left|\frac{\partial u}{\partial x}\right|^\rho \frac{\partial u}{\partial x} - f = h_0 \tag{12}$$

For resolve (10) and (11) we denotes.

$$\frac{\partial v}{\partial t} = v'; ...; \left|\frac{\partial v}{\partial t}\right|^{\rho} \frac{\partial v}{\partial t} = \beta(v'); L = -A$$

We begin the functional space V

$$V = \begin{cases} v : v \in L^2(0,T,H_0^1(\Omega)); v' \in L^2(0,T,H_0^1(\Omega)) \cap L^p(Q); \\ v'' \in L^2(0,T,L^2(\Omega)); \int_0^T v(t)dt = 0;\ v(0) = v(T);\ v'(0) = v'(T) \end{cases} \tag{13}$$

The Banach structure of V is defined by

$$\|v\|_V = \|v\|_{L^2(0,T,H_0^1(\Omega))} + \|v'\|_{L^2(0,T,H_0^1(\Omega))} + \|v'\|_{L^P(Q)} + \|v''\|_{L^2(0,T,L^2(\Omega))}$$

We define the bilinear form:

$$a\,(u,v) = \int_0^T [(u'' + u', v) + (Au, v) + (\beta(u'), v)]dt \tag{14}$$

The weak formulation of (10)and(11) is to find $u \in V$ such that

$$a\,(u,v) = \int_0^T (f, v')\forall v \in V \tag{15}$$

Following some ideas of Lions for obtain the elliptic regularization, given $\delta > 0$ and $u, v \in V$ we define

$$\begin{aligned} \pi_\delta\ (u,v) \ &= \ \delta \int_0^T [(u'', v'') + (u', v') + (Au', v')]ds + \\ &\int_0^T (u'' + u' + \mathrm{A}\,u + \beta(u'), v')]ds \end{aligned} \tag{16}$$

The application $v \longrightarrow \pi_\delta\,(u,v)$ is continuous on V so there existes an application $B_\delta \in V'$:

$$\pi_\delta\ (u,v) = (B_\delta\ (u), v) \tag{17}$$

The linear operator $B_\delta\ : V \longrightarrow V'$satisfies the four properties:

B_δ is bounded in V' for all bounded set in V and is a hemicontinuous and is a strictly monotonous and is coercive

In view of these properties and as consequence of theorem of Lions [5], there existe unique a function $u_\delta\ \in V$:

$$\pi_\delta\ (u_\delta\ , v) = \int_0^T (f, v')dt \quad \forall v \in V \tag{18}$$

2.1.2 Second step: a first priori estimates.

Explicitly the elliptic regularization (18) and setting $v = u_\delta$ we obtain

$$\delta \int_0^T \left[|u''_\delta\ |^2 + |u'_\delta\ |^2 + \|u'_\delta\ \|^2 \right] dt + \tag{19}$$

$$\int_0^T \left[|u'_\delta\ |^2 + (\beta(u'_\delta\ ), u'_\delta\ ) \right] dt = \int_0^T (f, u_\delta\ )\, dt$$

or

$$\int_0^T (\beta(u'), u')\, dt = \|u'\|^P_{L^P(Q)}\, and \int_0^T udt = 0 \Rightarrow \|u\|_{L^2(0,T,H_0^1(\Omega))} \leq c\, \|u'\|_{L^2(0,T,H_0^1(\Omega))}$$

Then

$$u'_\delta \text{ is bounded in } L^p(Q) \text{ when } \delta \to 0 \tag{20}$$

$$\delta \int_0^T \left[|u''_\delta\ |^2 + |u'_\delta\ |^2 + \|u'_\delta\ \|^2 \right] dt \leq c \tag{21}$$

or, $\int_0^T u_\delta\ dt = 0$ we have by (20) and (21) that :we have by

$$u_\delta \text{ is bounded in } L^p(Q) \tag{22}$$

$$\delta \int_0^T \|u_\delta\|^2 \, dt \leq c_1 \tag{23}$$

2.1.3 Third step: a second priori estimates.

Putting in (18) v

$$v(t) = \int_0^T u_\delta(s)ds - \frac{1}{T}\int_0^T (T-s)u_\delta(s)ds \tag{24}$$

$$\begin{cases} \displaystyle\int_0^T v dt = 0 \quad \forall v \in V \\ v' = u_\delta \end{cases} \tag{25}$$

Taking into account (24) in (18) we get

$$\begin{gathered} \delta \int_0^T [(u''_\delta, u'_\delta) + (u'_\delta, u_\delta) + (Au'_\delta, u_\delta)]dt + \int_0^T [(u''_\delta, u_\delta) + (u'_\delta, u_\delta) \\ + \|u_\delta\|^2 + (\beta(u'_\delta), u'_\delta)]dt = \int_0^T (f, u_\delta)\, dt \end{gathered} \tag{26}$$

By using periodicity of $u_\delta, u'_\delta \in V$, we obtain:

$$\int_0^T (u''_\delta, u'_\delta)dt = \int_0^T (Au'_\delta, u_\delta)dt = 0 \tag{27}$$

And

$$\begin{aligned} \int_0^T (u''_\delta, u_\delta)dt &= (u'_\delta(T), u_\delta(T)) - (u'_\delta(0), u_\delta(0)) - \\ \int_0^T (u'_\delta, u'_\delta)dt &= -\int_0^T |u'_\delta|^2 \, dt \end{aligned} \tag{28}$$

By (27), (28) and (20) we have

$$\left|\int_0^T (u_\delta'', u_\delta)dt\right| \leq c \text{ when } \delta \to 0 \tag{29}$$

Also, from (20) and (22) we obtain

$$\left|\int_0^T (\beta(u_\delta'), u_\delta)dt\right| \leq \|\beta(u_\delta')\|_{L^q(Q)} \|u_\delta\|_{L^p(Q)} \leq c' \tag{30}$$

Combining (27), (29), (30) with (26) we deduce

$$\int_0^T \|u_\delta\|^2 dt \leq C \tag{31}$$

3 Uniqueness of solution:

Theorem 2 *Under the hypotheses of the theorem of existence,we consider two solutions u_1 and u_2 of the problem (P) then $u_1 = u_2$*

Proof. *We subtract the equations* (9) *corresponding to u_1 and u_2 and sitting $\phi = u_1 - u_2$ we have* ■

$$\phi'' + \phi' + A\phi + \beta(u_1') - \beta(u_2') \tag{32}$$

Denoting by (η_δ) the regularizing sequence a similar argument by Brézis [3] we obtain

$$\phi' * \eta_\delta * \eta_\delta = \phi * \eta_\delta' * \eta_\delta \tag{33}$$

Hence, by using (5) and (6), we have

$$\phi = \varphi + \phi_0 : \ \ \phi_0 \in V \text{ and } \varphi \in L^2\left(0, T; H_0^1(\Omega)\right) \tag{34}$$

From (32); (33); (34); we get

$$\phi' * \eta_\delta * \eta_\delta = \phi * \eta'_\delta * \eta_\delta = \varphi' * \eta_\delta * \eta_\delta \tag{35}$$

show that

$$\int_0^T (\phi'', \phi' * \eta_\delta * \eta_\delta)dt$$

have sense and nul

$$\begin{aligned} \int_0^T \frac{d}{dt}(\phi', \phi' * \eta_\delta * \eta_\delta)dt &= \int_0^T (\phi'', \phi' * \eta_\delta * \eta_\delta)dt + \\ \int_0^T (\phi'', \phi' * \eta_\delta * \eta_\delta)dt &= 2\int_0^T (\phi'', \phi' * \eta_\delta * \eta_\delta)dt = 0 \end{aligned} \tag{36}$$

Therfore

$$\int_0^T (\phi'', \phi' * \eta_\delta * \eta_\delta)dt = \frac{1}{2}\int_0^T \frac{d}{dt}(\phi', \phi' * \eta_\delta * \eta_\delta)dt = 0 \tag{37}$$

$\phi'$and η_δ periodic then we have

$$\int_0^T (\phi, \phi' * \eta_\delta * \eta_\delta)dt = \int_0^T (\phi', \phi' * \eta_\delta * \eta_\delta)dt = \int_0^T (A\phi, \phi' * \eta_\delta * \eta_\delta)dt = 0 \tag{38}$$

From (32); (37); (38);we obtain

$$\int_0^T (\beta(u'_1) - \beta(u'_2), \phi' * \eta_\delta * \eta_\delta)dt = 0 \tag{39}$$

Passage to the limit in (39) we have

$$\int_0^T (\beta(u'_1) - \beta(u'_2), u'_1 - u'_2)dt = 0 \tag{40}$$

where

$$u_1' - u_2' = 0 \Rightarrow u_1' = u_2' \tag{41}$$

This implies that

$$\phi = u_1 - u_2 = \theta, \quad \theta \text{ independent of } t \tag{42}$$

From (38) and (42) we obtain

$$\int_0^T (A\theta, \theta)dt = 0 \qquad \forall \theta \in V \tag{43}$$

we deduce from (5)

$$\theta \in H_0^1(\Omega) + W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \tag{44}$$

By (42) and (44) we have the uniqueness of solution.

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