A Similar Nonlinear Telegraph Problem
Governed By Lamé System

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Abstract. Introducing the Lamé operator in the telegraph equation, we obtain theoretically a similar nonlinear system. In this work we are interested in the existence and uniqueness of function \(u = u(x,t), x \in \Omega, t \in (0,T)\) solution for the new system by the elliptic regularization method.

Keywords: Lamé system, Elliptic regularization, Monotone operators.

1 Notations and position of the problem

Let \(\Omega\) an open bounded domain of \(\mathbb{R}^n\), with regular boundary \(\Gamma\). We denote by \(Q\) the cylinder \(\mathbb{R}^n_x \times \mathbb{R}_t; Q = \Omega \times [0,T]\), with boundary \(\Sigma\). L designed Lamé system define by \(\mu \Delta + (\lambda + \mu) \nabla \text{div}, \lambda \) and \(\mu\) are constants Lamé with \(\lambda + \mu \geq 0\) and \(h,f\) are functions. We look for the existence and uniqueness of a function \(u = u(x,t), x \in \Omega, t \in [0,T]\), solution of the problem (P)

\[
\begin{cases}
  u'' + u' + u - Lu + |u'|^{p-2}u' = f & \text{in } Q \\
  u = 0 & \text{on } \Sigma \\
  u(x,0) = u(x,T) & \forall x \in \Omega \\
  u'(x,0) = u'(x,T) & \forall x \in \Omega
\end{cases}
\]

2 Existence of the solution

Theorem 1. Assume that \(\Omega\) is bounded open of \(\mathbb{R}^n\) are given \(f\), with \(f \in L(Q)\). Then there exists a function \(u = w_0 + w\) satisfying (P)

\[
\begin{align*}
  w_0 & \in H_0^1(\Omega) + W^{2,q}(\Omega) \cap W_0^{1,2}(\Omega) \\
  w & \in L^2(0,T; H_0^1(\Omega)) \\
  w' & \in L^p(Q)
\end{align*}
\]
Proof we use an approach due to G. Prodi [11] we have:

\[
\begin{align*}
\begin{cases}
    u = w_0 + w \\
    w_0 \text{ independent of } t
\end{cases}
\end{align*}
\]  

(1.5)

We introduce the Prodi idea (1. 5) in (1.1.1) we having

\[
\begin{align*}
u'' + u' + u - Lu + |u'|^{p-2}u' - f &= f + \\
Lu_0
\end{align*}
\]  

(1.6)

We consider the derivative of (1.6) we obtain

\[
\frac{d}{dt} \left( u'' + u' + u - Lu + |u'|^{p-2}u' \right) = \frac{df}{dt}
\]  

(1.7)

And

\[
\begin{align*}
\int_0^T u dt &= 0 \\
u(T) &= u(0) \\
u'(x, 0) &= u'(x, T)
\end{align*}
\]  

(1.8)

We deduce to (1.7)

\[
u'' - Lu + |u'|^{p-2}u' - f = h_0 \quad \text{with } h_0 \text{ independent of } t
\]  

(1.9)

For resolve (1.7) and (1.8) we denotes. \( L = -\lambda; \beta(u') = |u'|^{p-2}u' \)

And we define the functional space \( V \):

\[
V = \left\{ v \in L^2(0,T;H_0^1(\Omega)); \quad v \in L^2(0,T,(H_\lambda^1(\Omega)) \cap L^p(Q); \\
\int_0^T v(t)dt = 0; \quad v(T) = v(0); \quad v'(T) = v'(0)
\right\}
\]  

(1.10)

The Banach structure of \( V \) is defined by

\[
\|v\|_V = \|v\|_{L^2(0,T;H_0^1(\Omega))} + \|v'\|_{L^2(0,T,H_\lambda^1(\Omega))} + \|v\|_{L^p(Q)} + \|v\|_{L^2(0,T,L^2(\Omega))}
\]

We define the bilinear form:

\[
b(u, v) = \int_0^T \left( (u'', v') + (Au, v) + (\beta(u'), v) \right) dt
\]  

(1.11)

The weak formulation of (1.7) and (1.8) is to find \( u \in V \) such that

\[
b(u, v) = \int_0^T (f, v')dt \quad \forall \ v \in V
\]  

(1.12)

But (1.12) not coercive.

Then we following some ideas of Lions for obtain the elliptic regularization, given \( \delta > 0 \) and \( u, v \in V \), we define

\[
\pi_\delta(u, v) = \delta \int_0^T [(u'', v') + (Au, v')] ds + \int_0^T (u'' + Au + \beta(u'), v') ds.
\]  

(1.13)

The application \( v \rightarrow \pi_\delta(u, v) \) is continuous on \( V \) so there exists an application

\[
B_\delta \in \mathcal{V}: \pi_\delta(u, v) = (B_\delta(u), v)
\]  

(1.14)

The linear operator \( B_\delta : V \rightarrow V \) satisfies the four properties:

\( B_\delta \) is bounded in \( V \) for all bounded set in \( V \) and is a hemi continuous and is a strictly monotone and is coercive.
In view of these properties and as consequence of theorem of Lions [4], there exist unique a function $u_\delta \in V$:

$$\pi_\delta(u_\delta, v) = \int_0^T (f, v') dt \quad \forall v \in V$$  \hspace{1cm} (1.15)

### 2.1 A priori estimates I

Explicitly the elliptic regularization (1.15) and setting $v = u_\delta$, we obtain:

$$\delta \int_0^T [\|u''_\delta\|^2 + \|u'_\delta\|^2] dt + \int_0^T [(u'_\delta)^2 + (\beta(u'_\delta), u'_\delta)] dt = \int_0^T (f, u_\delta) dt$$  \hspace{1cm} (1.16)

Or

$$\int_0^T (\beta(u'), u') dt = \|u'\|_{L^p(\Omega)}^p \quad \text{And} \quad \int_0^T v dt = 0 \Rightarrow \|u\|_{L^2(0,T; H^1(\Omega))} \leq C \|u'\|_{L^2(0,T; H^1(\Omega))}$$

Then

$u'_\delta$ is bounded in $L^p(\Omega)$ when $\delta \to 0$  \hspace{1cm} (1.17)

$$\delta \int_0^T [\|u''_\delta\|^2 + \|u'_\delta\|^2 + \|u'_\delta\|^2] dt \leq C$$  \hspace{1cm} (1.18)

Or

$$\int_0^T u_\delta dt = 0.$$  \hspace{1cm} (1.19)

And

$$\delta \int_0^T \|u_\delta\|^2 dt \leq C$$  \hspace{1cm} (1.20)

### 2.2 A priori estimates II

Exchange in (1.15) $v$ with:

$$v(t) = \int_0^t u_\delta(s) ds - \frac{1}{T} \int_0^T (T-s)u_\delta(s) ds$$  \hspace{1cm} (1.21)

We verify that:

$$\begin{cases} 
\int_0^T v dt = 0 & \forall v \in V \\
0 & v' = u_\delta 
\end{cases}$$  \hspace{1cm} (1.22)

Taking into account (1.21) in (1.15) we get

$$\delta \int_0^T [(u''_\delta, u'_\delta) + (u'_\delta, u_\delta) + (Au'_\delta, u_\delta)] dt + \int_0^T [(u''_\delta, u_\delta) + (u'_\delta, u_\delta)] dt$$
By using periodicity of \( u_\delta, u'_\delta \in V \), we obtain:

\[
\int_0^T (u''_\delta, u'_\delta) dt = \int_0^T (Au'_\delta, u_\delta) dt = 0
\]  \hspace{1cm} (1.24)

And

\[
\int_0^T (u''_\delta, u_\delta) dt = (u'_\delta(T), u_\delta(T)) - (u'_\delta(0), u_\delta(0)) - \int_0^T (u'_\delta, u'_\delta) dt
\]

\[
= -\int_0^T |u'_\delta|^2 dt
\]  \hspace{1cm} (1.25)

By (1.24) and (1.17) we have

\[
\left| \int_0^T (u''_\delta, u_\delta) dt \right| \leq C \quad \text{when } \delta \to 0
\]  \hspace{1cm} (1.26)

Also, from (1.17) and (1.19) we obtain:

\[
\left| \int_0^T \beta(u'_\delta), u_\delta) dt \right| \leq \|\beta(u'_\delta)\|_{L^p(Q)} \|u_\delta\|_{L^p(Q)} \leq C'
\]  \hspace{1cm} (1.27)

Combining (1.24), (1.26), (1.27) with (1.23) we deduce

\[
\int_0^T \|u_\delta\|^2 dt \leq C
\]  \hspace{1cm} (1.28)

### 2.3 Passage to the limit

From (1.17) and (1.28) that there exists a subsequence from \((u_\delta)\), such that

\[
u_\delta \to 0 \quad \text{weak in } L^2(0, T; H_0^1(\Omega))
\]  \hspace{1cm} (1.29)

\[
u'_\delta \to \nu' \quad \text{weak in } L^p(Q)
\]  \hspace{1cm} (1.30)

\[
\beta(u'_\delta) \to \chi \quad \text{weak in } L^q(Q)
\]  \hspace{1cm} (1.31)

Passage to the limit in (1.15) we obtain

\[
\int_0^T [(-u', v') + (Au, v') + (\chi, v')] dt = \int_0^T (f, v') dt \quad \forall v \in V
\]  \hspace{1cm} (1.32)

Use the convolution technical in (1.32) we have
When

\[ \int_0^T (\chi, u'') dt = \int_0^T (f, u') dt \quad \forall v \in V \quad (1.34) \]

### 3 Uniqueness of solution:

**Theorem**

Under the hypotheses of the theorem of existence, we consider two solutions \( u_1 \) and \( u_2 \) of the problem (P) then \( u_1 = u_2 \).

**Proof:** We subtract the equations (1.9) corresponding to \( u_1 \) and \( u_2 \) and sitting

\[ \phi'' + A\phi + \beta(u_1) - \beta(u_2) \]

Denoting by \((\eta_\delta)\) the regularizing sequence a similar argument by Brézis [2] we obtain

\[ \phi' \ast \eta_\delta = \phi' \ast \eta_\delta = [\phi' \ast \eta_\delta] \quad (2.2) \]

Hence, by using (1.2) and (1.3), we have

\[ \phi = \varphi + \phi_0 : \phi_0 \in V \text{ and } \varphi \in L^2(0,T,H_0^1(\Omega)) \quad (2.3) \]

From (2.2) we get

\[ \phi' \ast \eta_\delta = \phi' \ast \eta_\delta = \psi' \ast \eta_\delta \quad (2.4) \]

Show that

\[ \int_0^T (\psi', \phi' \ast \eta_\delta) dt = 0 \]

When

\[ \int_0^T \frac{d}{dt} (\phi', \phi' \ast \eta_\delta) dt \]

\[ = \int_0^T (\psi', \phi' \ast \eta_\delta) dt + \int_0^T (\phi'', \phi' \ast \eta_\delta) dt = 0 \quad (2.5) \]

Therefore

\[ \int_0^T (\phi'', \phi' \ast \eta_\delta) dt = \frac{1}{2} \int_0^T \frac{d}{dt} (\phi', \phi' \ast \eta_\delta) dt = 0 \quad (2.6) \]

\( \phi \) and \( \eta_\delta \) periodic then we have

\[ \int_0^T (\phi, \phi' \ast \eta_\delta) dt = \int_0^T (\psi', \phi' \ast \eta_\delta) dt = \int_0^T (A\phi, \phi' \ast \eta_\delta) dt \quad (2.7) \]

From (2.1); (2.6) and (2.7) we obtain:
Passage to the limit in (2.8) we have
\[ \int_{0}^{T} \left( \beta(u'_1) - \beta(u'_2), \phi' \right)_{\eta_{\theta}} = 0 \] (2.8)
Where
\[ u'_1 - u'_2 = 0 \Rightarrow u'_1 = u'_2 \] (2.10)
This implies that
\[ \phi = u_1 - u_2 = 0, \theta \text{ independent of } t \] (2.11)
\[ \text{From (2.7) and (2.11) we obtain} \]
\[ \int_{0}^{T} (A\theta, \theta)_{\mathcal{D}} = 0 \quad \forall \theta \in \mathcal{D} \] (2.12)
We deduce from (1.2)
\[ \theta \in H_0^1(\Omega) + W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \] (2.13)
By (2.12) and (2.13) and using theorem of Green we have \( (A\theta, \theta) = 0 \Rightarrow \theta = 0 \)
Where the uniqueness of solution.

References